Open problems related to polynomial interpolation

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Abstract: A lot of recent research has been motivated by old still open problems involving polynomial interpolation. I'll survey some of these open problems, and say a little about what they have led to.

Slides available eventually at my website (green text is clickable): https://unlblh.github.io/BrianHarbourne/

(Set up: $k = \overline{k}$) Ideals of fat points

Given $p_1, \ldots, p_r \in \mathbb{A}^n \subset \mathbb{P}^n$, $m_1, \ldots, m_r \geq 0$.

$I = \cap_{i=1}^r I(p_i)^{m_i} \subseteq k[\mathbb{A}^n] = k[t_1, \ldots, t_n]$.

Homogenization: $I^* = J = \cap_{i=1}^r J(p_i)^{m_i} \subseteq k[\mathbb{P}^n] = k[x_0, x_1, \ldots, x_n]$.

$V_d(I) = \langle f \in I : \deg f \leq d \rangle$.

$W_d(J) = \langle F \in J : \deg f = d \rangle = [J]_d$.

$h_J(d) = \dim W_d(J)$ (Hilbert function of J).

Why ideals of fat points?

Easy Fact: $\dim_k V_d(I) = h_J(d) = h^0(X, \mathcal{L}(d, m_1 p_1, \ldots, m_r p_r))$ where $\pi : X \to \mathbb{P}^n$ is the blow up at the points p_i, and where $\mathcal{L}(d, m_1 p_1, \ldots, m_r p_r) = \mathcal{O}(d) \otimes \bigotimes_i (\pi^{-1}(p_i)^{\otimes - m_i})$ is a certain line bundle on X.

Warm up: $n = 1$

Find the Hilbert function h_J of J in terms of the m_i when $n = 1$.

I.e., find $h_J(d) = \dim_k W_d(J)$ for all d.

Answer: $h_J(d) = \max(0, d + 1 - \sum m_i)$

Thus h_J does not depend on the location of the points.

Reason: $k[\mathbb{A}^1]$ is a PID.

$n > 1$: Some background

Now h_J depends on the relative location of the points.

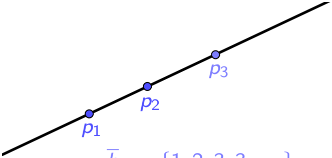
Definition (h-vector): $\overline{h}_J = h_{R/J} = h_R - h_J$, where $R = k[\mathbb{P}^n]$.

($\overline{h}_J(d) = \#$ lin. indep. vanishing conditions imposed in degree d.)

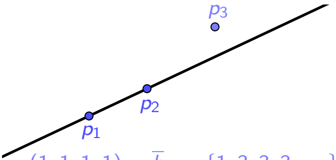
Let $\mu(d; m_1, \ldots, m_r) = \max \overline{h}_J(d)$ over all choices of distinct p_i.

Easy fact: The maximum is achieved for general points p_i (i.e., for $(p_1, \ldots, p_r)$ in a nonempty open subset $U \subset (\mathbb{P}^n)^r$).

Example: 3 points $p_1, p_2, p_3 \in \mathbb{P}^2$, $m_i = 1$:



$\overline{h}_J = \{1, 2, 3, 3, \ldots\}$



$\mu(1; 1, 1, 1) = \overline{h}_J = \{1, 3, 3, 3, \ldots\}$

3 points impose 3 conditions but sometimes only 2 are lin. indep.

Single point case: $r = 1$; $p = p_1$, $m = m_1$

We may assume $p = (0, \dots, 0) \in \mathbb{A}^n$.

Here $I = I(p)^m = (t_1, \dots, t_n)^m$ is a monomial ideal:

$$t_1^{s_1} \cdots t_n^{s_n} \in V_d(I) \text{ if and only if } m \leq s_1 + \cdots + s_n \leq d.$$

Thus:

$$h_J(d) = \dim W_d(J) = \dim V_d(I) = \min(0, \binom{d+n}{n} - \binom{m+n-1}{n}).$$

I.e., $\overline{h}_J(d) = \min(\binom{d+n}{n}, \binom{m+n-1}{n})$:

think of this as saying a point of multiplicity m imposes $\binom{m+n-1}{n}$ linear conditions on the $\binom{d+n}{n}$ forms of degree d, where here the conditions are as independent as possible.

Any n, any r points p_i, any multiplicities m_i

Corollary: $h_J(d) \geq \max(0, \binom{d+n}{n} - \sum_i \binom{m_i+n-1}{n})$

Question: Does "=" hold if the points are general?

Answer: Yes, if $m_i = 1$ for all i. But not in general otherwise.

Open Problems

Open Problem 1: When does

$$h_J(d) = \max(0, \binom{d+n}{n} - \sum_i \binom{m_i+n-1}{n})$$

fail if the points are general?

Example: Take $p_1, p_2 \in \mathbb{P}^2$, $m_1 = m_2 = 2$:

$$1 = h_J(2) > 0 = \max(0, \binom{2+2}{2} - \binom{3}{2} - \binom{3}{2}).$$

(N.B. If F defines the line through p_1, p_2, then $F^2 = \gcd(W_2(J))$.)

Open Problem 2: Find $h_J(d)$ in terms of $d, m_1, \ldots, m_r, n$, assuming the points are general.

There are conjectures only for $n = 2$ (SHGH Conjecture) and $n = 3$ (Laface-Ugaglia Conjecture).

SHGH: $n = 2$

Conjectures were give by:

Segre 1960, Harbourne 1986, Gimigliano 1987, Hirschowitz 1989 (all equivalent!: Ciliberto, Miranda (2001))

Segre's Conjecture: If equality fails, then $\gcd(W_d(J))$ is not squarefree.

(Compare to the example above.)

The others use results of Nagata to reduce the problem to:

Conjecture: if $m_1 \geq \cdots \geq m_r \geq 0$, $d \geq m_1 + m_2 + m_3$, then "$=$" holds.

Nagata's (Still Open) Conjecture: about 1960

Assume $r \geq 10$, $m_i = m$ for all i and the r points p_i are general.

Conjecture (Nagata): If $h_J(d) > 0$, then $d > m\sqrt{r}$.

Comments:

1. Nagata proved this when $\sqrt{r} \in \mathbb{Z}$ and used it to give a counterexample to Hilbert's 14th Problem.

2. The SHGH Conjecture implies the Nagata Conjecture.

Nagata's Conjecture is about "symbolic powers"

Given $J = \cap_{i=1}^{r} J(p_i)^{m_i}$; denote $\cap_{i=1}^{r} J(p_i)^{mm_i}$ by $J^{(m)}$.

Definition: $J^{(m)}$ is called the mth symbolic power of J.

For general p_i and $J = \cap_{i=1}^{r} J(p_i)$, Nagata's Conjecture posits

$$h_{J^{(m)}}(d) > 0 \implies d > m\sqrt{r}.$$

Fun Fact: Let $I = \cap I(p_i)^{m_i}$ and $J = \cap_{i=1}^{r} J(p_i)^{m_i}$. Then:

$I^m = I(p_1)^{mm_1} \cdots I(p_r)^{mm_r} = \cap_{i=1}^{r} I(p_i)^{mm_i}$ (Chinese Rem Thm) and

$J^m \subseteq J^{(m)}$, but typically

$J^m = J^{(m)}$ fails.

What goes wrong? We only get $J^m \subseteq J^{(m)}$:

$J^m = J^{(m)} \cap Q$ where Q is M-primary for $M = (x_0, \ldots, x_n)$.

The resurgence

Let $J = \cap_{i=1}^{r} J(p_i)^{m_i}$ for any r distinct points. Then $J^{(mn)} \subseteq J^m$

for all m (Ein, Lazarfsfeld, Smith, Hochster, Huneke: early 2000s).

Definition (resurgence: 2010 (Bocci-Harbourne)): For $J \neq (1)$, define

$$\rho(J) = \sup\{\frac{m}{s} | J^{(m)} \not\subseteq J^s\}.$$

Fact: $1 \leq \rho(J) \leq n$.

Proof: $\frac{m}{s} < 1 \implies \frac{m}{s} \leq \rho(J)$ hence $1 \leq \rho(J)$:
$m < s \implies J^m \not\subseteq J^s \implies J^{(m)} \not\subseteq J^s \implies \frac{m}{s} \leq \rho(J)$.

$\frac{m}{s} \geq n \implies m \geq ns \implies J^{(m)} \subseteq J^{(sn)} \subseteq J^s \implies \rho(J) \leq n$. □

What can be said about J for extremal values of $\rho(J)$?

Open Problem: When is $\rho(J) = 1$?

Open Problem: Classify all fat point ideals J with $\rho(J) = 1$.

If $J^{(m)} = J^m$ for all m then $\rho(J) = 1$.

Theorem (Harbourne, Kettinger, Zimmitti: 2022): Assume J is reduced (i.e., $m_i = 1$ for all i). Then TFAE:

(1) J is a complete intersection;

(2) $J^m = J^{(m)}$ for all m; and

(3) $\rho(J) = 1$.

Open Question: What about when J is not reduced (i.e., when $m_i > 1$ for some i)?

Open Problem: is $\rho(J)$ ever equal to n?

Can we improve on ELS, HH; i.e., on $J^{(mn)} \subseteq J^m$ for all m?

Containment Conjecture 1 (Harbourne: 2008): $J^{(mn-n+1)} \subseteq J^m$ holds for all m. (Often true but not always.)

Containment Conjecture 2 (Harbourne-Huneke: 2011): $J^{(mn)} \subseteq M^{m(n-1)} J^m$ holds for all m. (Open.)

If Containment Conjecture 2 is true, then so are certain conjectures of Chudnovsky and Demailly.

Let's not give up on Containment Conjecture 1!

Grifo Conjecture: $J^{(mn-n+1)} \subseteq J^m$ for $m \gg 0$. I.e., $J^{(mn-n+1)} \subseteq J^m$ fails for only finitely many m.

Any counterexample to GC has $\rho(J) = n$.