Powers versus Symbolic Powers of Ideals

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Outline

What is a symbolic power?

Given a homogeneous ideal: $I \subseteq k[\mathbf{P}^N] = k[x_0, \ldots, x_N] = R$.

Remove foreign primary components from I^m to get $I^{(m)}$:

Definition (mth Symbolic Power)

$I^{(m)} = R \cap (\cap_{P \in \mathrm{Ass}(I)} I^m R_P)$

Example: Let $I = I(p_1) \cap \cdots \cap I(p_n)$ where $p_1, \ldots, p_n \in \mathbf{P}^N$

and $I(p_i)$ = ideal in R of forms vanishing at p_i:

$I^m = (I(p_1) \cap \cdots \cap I(p_n))^m$ Ordinary Power

$I^{(m)} = I(p_1)^m \cap \cdots \cap I(p_n)^m$ Symbolic Power

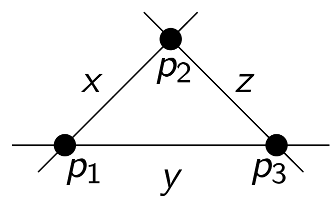
How do I^m and $I^{(m)}$ compare?

Fact: $I^m \subseteq I^{(m)}$

Example: $I^m = I^{(m)}$ can fail.

$I = I(p_1, p_2, p_3)$ is generated in degree 2 so

$xyz \notin I^2$, but $xyz \in I^{(2)}$, hence $I^2 \subsetneq I^{(2)}$.



In terms of a useful measurement tool, α:

$\alpha(I^{(2)}) = 3$ but $\alpha(I^2) = 2\alpha(I) = 4$, where $\alpha(I)$

is the degree of a homogeneous generator of I of least degree.

Waldschmidt and Skoda (1976)

Waldschmidt: Asymptotic Behavior of $\alpha(I^{(m)})$

$I = I(p_1, \ldots, p_n) \subset \mathbf{C}[\mathbf{P}^N]$ $\qquad \alpha(I^{(i+j)}) \leq \alpha(I^{(i)}) + \alpha(I^{(j)})$

$$-2+ \frac{\alpha(I)}{N} \leq \overbrace{\lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}}^{\gamma(I)} \leq \frac{\alpha(I^{(m)})}{m} \leq \alpha(I)$$

Skoda (**) hard pretty easy (α is sublinear) easy

Proof of (**): Uses complex analytic methods.
Uses refinements of results of Bombieri on plurisubharmonic functions.

Alternate Proof of (**)

Recall:

Theorem (Ein-Lazarsfeld-Smith 2001 / Hochster-Huneke 2003)

Let $I \subseteq k[\mathbf{P}^N]$ be a homogeneous ideal and $m > 0$. Then

$$I^{(mN)} \subseteq I^m.$$

Proof: Find an ideal J such that $I^{(mN)} \subseteq J \subseteq I^m$.
ELS: uses multiplier ideals
HH: uses Frobenius powers and tight closure

Alternate Proof of (**): $I^{(mN)} \subseteq I^m \Rightarrow$

$$\frac{\alpha(I)}{N} = \frac{m\alpha(I)}{mN} = \frac{\alpha(I^m)}{mN} \leq \frac{\alpha(I^{(mN)})}{mN} \underset{m\to\infty}{\longrightarrow} \gamma(I)$$

Are W-S & ELS/HH Optimal?

Let $0 \neq I \subsetneq k[\mathbf{P}^N]$ be homogeneous.

W-S inequality: $\frac{\alpha(I)}{\gamma(I)} \leq N$. ELS-HH result: $I^{(m)} \subseteq I^r$ if $N \leq \frac{m}{r}$.

Question: Are the results of W-S & ELS/HH optimal?

Can the constant N be universally decreased?

Theorem (Bocci-H___ JAG 2009): W-S is optimal: $\sup\{\frac{\alpha(I)}{\gamma(I)}\} = N$ where the sup is over ideals of finite sets of points in $\mathbf{P}^N$.

If W-S is optimal, so is ELS-HH:

Lemma (Bocci-H___ JAG '09): $\frac{m}{r} < \frac{\alpha(I)}{\gamma(I)} \Rightarrow I^{(mt)} \not\subseteq I^{rt}$ for $t \gg 0$.

Nonetheless, can ELS-HH be improved?

Takagi & Yoshida give examples with $I^{(rN-1)} \subseteq I^r$.

Question (Huneke 2003)

For $I = I(p_1, \dots, p_n) \subset k[\mathbf{P}^2]$, is it true that $I^{(3)} \subseteq I^2$?

Bocci-H___: $\exists\ I \subseteq k[\mathbf{P}^N]$ with $I^{(rN-N)} \not\subseteq I^r$ for $r \gg 0$.

Fact (H___): For these ideals we have $I^{(rN-(N-1))} \subseteq I^r$.

This suggests:

Conjecture (H___ 2008)

Let $I \subseteq k[\mathbf{P}^N]$ be homogeneous. Then $I^{(rN-(N-1))} \subseteq I^r$.

Evidence supporting conjecture

Example 1: If $I = I(p_1)^{m_1} \cap \cdots \cap I(p_n)^{m_n} \subset k[\mathbf{P}^N]$ is a fat point ideal, where $\mathrm{char}(k) = q > 0$, and $r = q^i$, then the conjectural containment $I^{(rN-(N-1))} \subseteq I^r$ holds.

Example 2: The conjecture holds for monomial ideals in all characteristics.

Example 3: The conjecture holds for the ideals which Bocci-H___ use to show optimality.

Example 4: The conjecture holds for all sets of generic points in $\mathbf{P}^2$, and also for generic points in $\mathbf{P}^N$ when the number of points is sufficiently large.

The Containment Problem

Consider homogeneous ideals $0 \neq I \subset k[\mathbf{P}^N]$.

An Old Problem

Which ideals have $I^{(r)} = I^r$ for all r?
(Answers given by Macaulay, Hochster, Morey, Li-Swanson, etc.)

A More General Problem

For each I, find all m and r with $I^{(m)} \subseteq I^r$.

Theorem (Bocci-H___ PAMS 2009)

Assume $\alpha(I) = \mathrm{reg}(I)$ *and* $\dim Z(I) = 0$. *Then*

$$I^{(m)} \subseteq I^r \text{ if and only if } \alpha(I^r) \leq \alpha(I^{(m)}).$$

Examples

Example 1: Let $I = I(p_1, \ldots, p_6)$, for $p_i \in \mathbf{P}^2$ general. Then

$$\alpha(I^{(m)}) = \left\lceil \frac{12m}{5} \right\rceil \text{ and } \alpha(I) = \mathrm{reg}(I) = 3,$$

so the Theorem applies and

$$I^{(m)} \subseteq I^r \text{ if and only if } m \geq \tfrac{5}{4}r - \tfrac{5}{12},$$

hence $I^{(2r-1)} \subseteq I^r$ for all r.

More examples

Example 2: Let n be a square such that $n = \binom{s+2}{2}$ for some s. There are infinitely many such squares; e.g.,

$$s = 0, 7, 48, 287, 1680, 9799, \ldots.$$

Let $I = I(p_1, \ldots, p_n)$, $p_i \in \mathbf{P}^2$ general. Then $\alpha(I) = \mathrm{reg}(I) = s + 1$, so the Theorem applies and

$$I^{(m)} \subseteq I^r \text{ iff } rs \leq 1 + \left\lfloor \frac{-3 + \sqrt{4n(m^2 + m) + 1}}{2} \right\rfloor,$$

hence $I^{(2r-1)} \subseteq I^r$ for all r.

More examples

Example 3: Let $I = I(p_1, \ldots, p_8)$, $p_i \in \mathbf{P}^2$ general.

Then $3 = \alpha(I) < \mathrm{reg}(I) = 4$ so Theorem does not apply.

By geometrical methods (Bocci-H___ PAMS 2009):

$$I^{(m)} \subseteq I^r \text{ iff either } m = r = 1 \text{ or } m \geq \frac{17}{12}r - \frac{1}{3},$$

hence $I^{(2r-1)} \subseteq I^r$ for all r.

Thank You.