The concept of geproci subsets of $\mathbb{P}^3$: a timeline

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Organizers: Susan Cooper and Sarah Mayes-Tang

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Slides will be available at my website:
`https://www.math.unl.edu/~bharbourne1/`

Timeline (in years before present)

$t = -95$ Grete Hermann, Emmy Noether's 1st student receives PhD (her 1926 thesis laid foundation for computer algebra)

$t = -89$ John von Neumann, proved impossibility of hidden variables in quantum mechanics in 1932

$t = -55$ John Stewart Bell showed von Neumann's proof did not show what was claimed (Bell's 1966 Theorem)

$t = -47$ In 1974 Max Jammer pointed out Hermann had in 1935 already raised the issue Bell addressed (but was largely ignored)

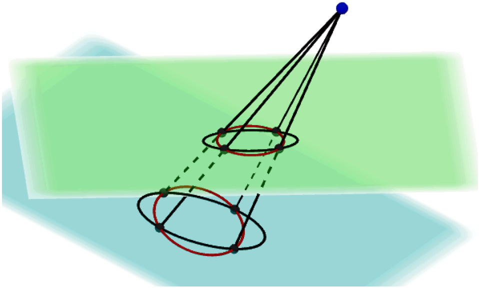
$t = -10$ A question is posted on Math Overflow.

$t = -10$: A question (6-8-2011).

A "general projection" means projection from a general point.

Let $Z \subset \mathbb{P}^3$ be a finite set of points. We say Z is (a, b)-GEPROCI if its GEneral PROjection to a plane is a Complete Intersection of curves of degrees a and b (with $a \leq b$).

Trivial example: A complete intersection (CI) of two curves in the same plane projects isomorphically to its image, so is trivially a CI.

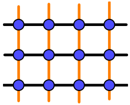


Mathoverflow Quest. 67265 by Francesco Polizzi: Are there nontrivial geproci sets Z?

$t = -9.99988$: Answer by Dmitri Panov (6-8-2011): Grids!

An (a, b)-*grid* is (a, b)-geproci. What is an (a, b)-grid?

It is given by a skew black lines and b skew orange lines, such that each black line meets each orange line in one point. The ab points form the grid. The lines are called *grid lines*.



A (3, 4)-grid

Based on Panov's construction, Polizzi edited his question:

(1) Are there nontrivial nongrid geproci sets?

(2) Can we classify them (at least for small numbers of points)?

Unexpected examples

Unexpectedly the answer to both questions is Yes, based on work on unexpected hypersurfaces. Let P be a general point and $Z \subset \mathbb{P}^n$ a finite set.

Number of degree d hypersurfaces containing Z with multipliciity m at P: $N(Z, d, m) = \dim[I(Z) \cap I(P)^m]_d$.

Expected number: $E(Z, d, m) = \max(0, \dim[I(Z)]_d - \binom{n+m-1}{n})$.

The hypersurfaces are unexpected if $N(Z, d, m) > E(Z, d, m)$.

If $d = m$, the hypersurfaces are cones.

So, where do you look for such Z?

t=-3.5 HMNT: H___, Migliore, Nagel and Teitler extended unexpectedness to hypersurfaces (preprint: arXiv:1805.10626; appeared as *Unexpected hypersurfaces and where to find them*, Mich. Math. J., 2021).

HMNT: look at root systems!

Given a root system $R \subset \mathbb{C}^{n+1}$, let $Z_R \subset \mathbb{P}^n$ be its projectivization. HMNT found a range of Z_R with unexpected hypersurfaces.

The following are the ones HMNT found which are cones in $\mathbb{P}^3$:

$R = D_4$: $Z_R \subset \mathbb{P}^3$ has unexpected cones of degrees 3 and 4.

$R = F_4$: $Z_R \subset \mathbb{P}^3$ has unexpected cones of degrees 4, 5, 6 and 7.

$R = H_4$: $Z_R \subset \mathbb{P}^3$ has unexpected cone of degree 6 (later P. Fraś, M. Zięba, arXiv:2107.08107, showed it had another one of degree 10).

$t = -3$: Workshop at Levico Terme in 2018

A working group at Levico Terme noticed something interesting:

$R = D_4$: $|Z_R| = 12$ has unexpected cones of degrees 3 and 4.

$R = F_4$: $|Z_R| = 24$ has unexpected cones of degrees 4 and 6.

Fact (Workshop working group at Levico Terme, 2018): Let $Z \subset \mathbb{P}^3$ be a finite set of points. If $|Z| = ab$ has unexpected cones of degrees a and b with no components in common, then Z is (a, b)-geproci.

Results of working group are written up in the appendix to:

CM: Chiantini, Migliore, "Sets of points which project to complete intersections," TAMS 374 (2021) 2581–2607 (arXiv:1904.02047).

This paper also gave results on classification:

Theorem (CM) All nontrivial nongrid geproci sets have at least 12 points (because nontrivial (a, b)-geproci sets with $2 = a \leq b$ or $a = b = 3$ are grids).

The 2018 Levico Terme working group

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Luca
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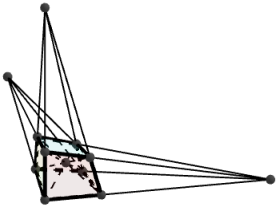
Let's look at $R = D_4$.

Z_{D_4} has 12 points and is $(3, 4)$-geproci . The 12 points come from a cube in 3 point perspective. It is contained in an unexpected cubic cone and an unexpected quartic cone.



The quartic cone is easy to see. It is the cone with vertex P on 4 skew lines containing Z_{D_4}.

The cubic cone comes from a pencil defined by two cubic cones. Here are the two cubic cones. And here is the pencil of cubic cones).

$t = -1$: More examples and a start on classification

New examples announced at MFO workshop, October, 2020:

Example (P. Fraś, M. Zięba, arXiv:2107.08107): Z_{H_4} is a nontrivial, nongrid $(6, 10)$-geproci.

Example (P. Pokora, T. Szemberg, J. Szpond, arXiv:2010.08863): A 60 point set due to Klein is a nontrivial, nongrid $(6, 10)$-geproci.

$t < -1$: The Geproci Squad, results and questions

Levico and the 10-2020 MFO workshop led to forming the Geproci Squad to work on geproci questions. Here is some work in progress.

(1) **Theorem** (Geproci Squad): Given $4 \leq a \leq b$, there is a nontrivial nongrid (a, b)-geproci set $Z \subset \mathbb{P}^3$.

(2) **Theorem** (Geproci Squad) Z_{D_4} is the unique nontrivial nongrid $(3, b)$-geproci set.

Some Questions:

(Q1) Which a, b have a unique nontrivial nongrid (a, b)-geproci Z? Is $a = 3, b = 4$ (i.e., Z_{D_4}) the only one?

(Q2) We know trivial geproci sets and $(2, b)$-grids ($b \geq 2$) do not come from unexpected cones. Do all other (a, b)-geproci Z come from unexpected cones of degrees a and b?

(Q3) What other nongrid geproci sets are there?

$t = -.0833$: Late breaking news! (11-3-2021)

Some time ago Squad member Giuseppe Favacchio ran across the fact that Z_{D_4} had been used in giving proofs of Bell's Theorem.

November 3, 2021: So Giuseppe searched further and found Z_{F_4} and Z_{H_4} also had been used in giving proofs of Bell's Theorem. And moreover, yet another set of points, based on the Penrose Dodecahedron, was used to prove Bell's Theorem. It's a 40 point set which also turned out to be geproci: it is a nontrivial, nongrid $(5, 8)$-geproci set.

New larger question: What exactly is the connection to Quantum Mechanics?

The Geproci Squad

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