Lefschetz properties, unexpectedness and geprociness

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Organizers: Emilia Mezzetti, Karim Adiprasito, Roberta Di Gennaro, Sara Faridi, Satoshi Murai & Uwe Nagel

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Slides available later today at my website (green text is clickable):
https://www.math.unl.edu/~bharbourne1/

Main reference: arXiv:2209.04820

This talk is on joint work of the Geproci Team (GT):

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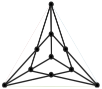
POLITUS: POLand, ITaly and the US

An international collaboration of 7 researchers whose logo is a stylized D_4 configuration:

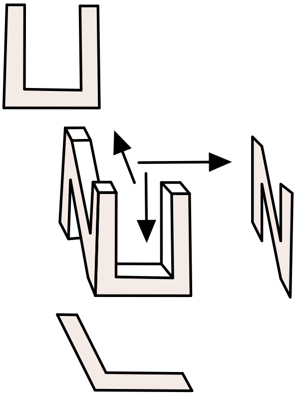


Abstract: AG Problems Motivated by Inverse Scattering

Studying inverse scattering problems has led to remarkable advances in scientific knowledge. Here we propose carrying this idea over to classification problems in algebraic geometry (AG).

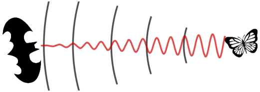
Inverse scattering Problems (ISP): try to discern structure from projected or reflected data.

Idea: classify structures algebro-geometrically based on properties of projected images.

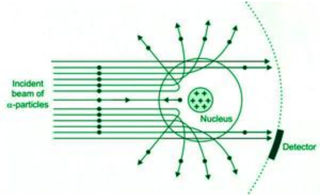


Some examples of ISP

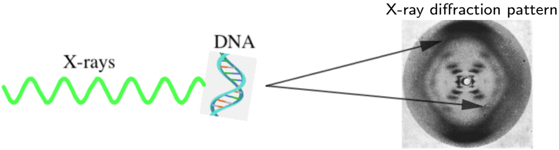
Echolocation (biology):



Rutherford scattering (physics; led to Bohr model of atom):



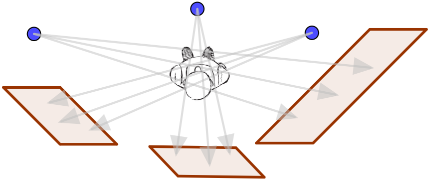
X-ray crystallography (chem/bio; led to DNA double helix model):



Rosalind Franklin
“Photo 51”

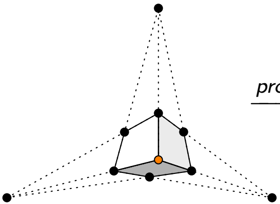
More examples

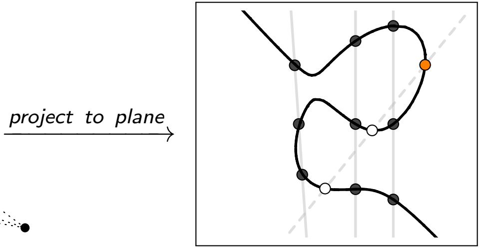
Tomography (medicine):



GePro-$\mathcal{P}$ (math): Pick a property $\mathcal{P}$ and classify finite point sets in $\mathbb{P}^n$ whose Ge neral Pro jections to a hyperplane satisfy $\mathcal{P}$.

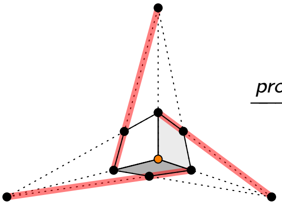
Here are 12 points (10 visible) in space whose projections from general points to a plane are complete intersections (so $\mathcal{P}$ is "being a CI"). These 12 points (known as D_4) are "geproci."



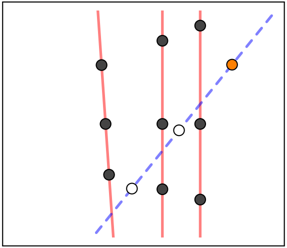


Why D_4 is geproci

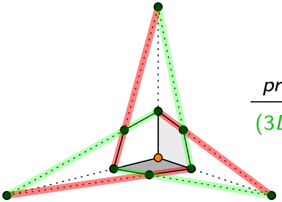
The quartic comes from lines through collinear points:



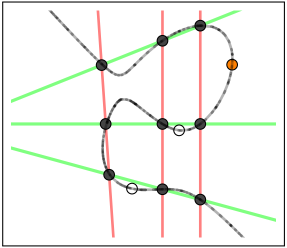
project to plane $\longrightarrow$



The cubic is one in a pencil of cubics:



project to plane $\longrightarrow$
(3*D* *visualization*)



Gepro-$\mathcal{P}$

General Problem: Given a property $\mathcal{P}$ of finite point sets $\overline{Z} \subset \mathbb{P}^{n-1}$, classify all finite $Z \subset \mathbb{P}^n$ such that $\overline{Z} \subset H \cong \mathbb{P}^{n-1}$ has property $\mathcal{P}$ (where $\overline{Z}$ is the image of Z under projection $\mathbb{P}^n \dashrightarrow H$ from a general point P to a hyperplane $H \subset \mathbb{P}^n$).

Example 1: Say $\mathcal{P}$ means "$\overline{Z}$ is Gorenstein". Then a set Z of $n+1$ general points in $\mathbb{P}^n$ is gepro-$\mathcal{P}$ since the image $\overline{Z}$ is a set of $n+1$ general points in H, which is Gorenstein.

Open Problem 1: Classify gepro-Gorenstein sets Z.

Every geproci set is also gepro-Gorenstein but not conversely.

Open Problem 2: Classify geproci sets in $\mathbb{P}^3$.

History of geproci

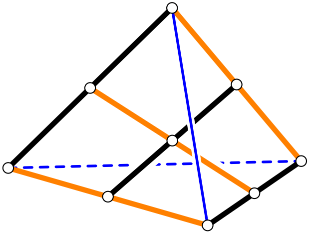
We know no interesting examples of geproci sets in $\mathbb{P}^n$ for $n > 3$.

We say $Z \subset \mathbb{P}^3$ is (a, b)-geproci if $\overline{Z}$ is the intersection of a curve A of degree a with a curve B of degree b, with $a \leq b$.

A geproci set Z in a plane $H \subset \mathbb{P}^3$ is called *degenerate*; it is just the complete intersection of two curves in H.

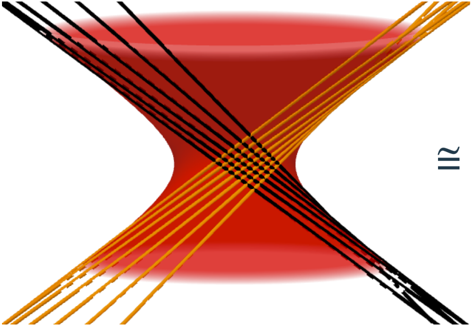
Question 1 (F. Polizzi 2011): Is every geproci set in $\mathbb{P}^3$ degenerate?

Answer (D. Panov, 2011): No!
(a, b)-grids are nondegenerate and geproci. I.e., $2 \leq a \leq b$ with A being a skew black lines and B being b skew orange lines, where each black line intersects each orange line in exactly 1 point. (Here $a = b = 3$.)

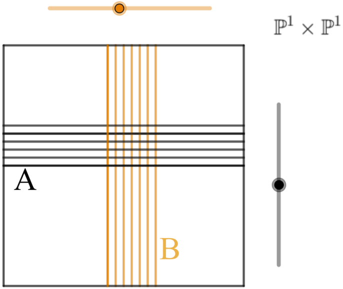


We understand grids.

Fact: For an (a, b)-grid with $3 \leq a \leq b$, the grid lines come from the rulings on a smooth quadric.



$\cong$



Fact: A $(2, b)$-grid consists of b points on each of two skew lines (but the grid lines need not all lie on a smooth quadric).

New Question and a partial answer

For simplicity, call a geproci set in $\mathbb{P}^3$ *trivial* if it is either a grid or contained in a plane.

Question 1′ (F. Polizzi 2011): Is every geproci $Z \subset \mathbb{P}^3$ trivial? If not, can such Z be classified up to projective equivalence, at least when $|Z|$ is small?

Answer (2018, Lefschetz Working Group at Levico Terme): Certain finite sets Z given by root systems (such as D_4 and F_4) which have unexpected cones (see Harbourne-Migliore-Nagel-Teitler: arXiv:1805.10626, Michigan Math. J. 2020) turn out to be nontrivial geproci sets.

Theorem (Levico Terme Working Group, 2018): A finite set $Z \subset \mathbb{P}^3$ is (a, b)-geproci if $|Z| = ab$ and it has unexpected cones of degrees $a \leq b$ with no common components.

The 2018 Levico Terme Working Group (LTWG)

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Unexpected cones are equivalent to WLP failing

Definition: A finite set of points $Z \subset \mathbb{P}^n$ satisfies $C(t)$ if it has an "unexpected cone" of degree t.

Taking Macaulay duals, $Z = \{p_1, \ldots, p_s\}$ satisfies $C(t)$ iff the Weak Lefschetz Property (WLP) fails for Z in degree $t-1$.

(For a point $Q = [q_0 : \cdots : q_n] \in \mathbb{P}^n$, let $L_Q = q_0x_0 + \cdots + q_nx_n$.)

WLP failing for Z in degree $t-1$ means $\times L_P$ does not have maximal rank:

$$\left[\frac{R}{(L_{p_1}^t, \ldots, L_{p_s}^t)}\right]_{t-1} \xrightarrow{\times L_P} \left[\frac{R}{(L_{p_1}^t, \ldots, L_{p_s}^t)}\right]_t \twoheadrightarrow \left[\frac{R}{(L_{p_1}^t, \ldots, L_{p_s}^t, L_P)}\right]_t$$

A result and an Open Problem

Theorem (Chiantini-Migliore, arXiv:1904.02047, TAMS 2021):
Every (a, b)-grid with $3 \leq a \leq b$ satisfies both $C(a)$ and $C(b)$.

Open Problem 3: Does every nontrivial (a, b)-geproci $Z \subset \mathbb{P}^3$ satisfy both $C(a)$ and $C(b)$?

D_4 played a special role

Theorem (Chiantini-Migliore, arXiv:1904.02047, TAMS 2021): The least $|Z|$ for a nontrivial geproci set is $|Z| = 12$. An example is given by the D_4 configuration of 12 points; it is $(3, 4)$-geproci.

Theorem 1 (GT, 2022): The D_4 configuration is, up to projective equivalence, the only nontrivial (a, b)-geproci set in $\mathbb{P}^3$ with $a \leq 3$.

D_4 motivated the following theorem:

Theorem 2 (GT, 2022): For each $4 \leq a \leq b$, there is a nontrivial (a, b)-geproci $Z \subset \mathbb{P}^3$.

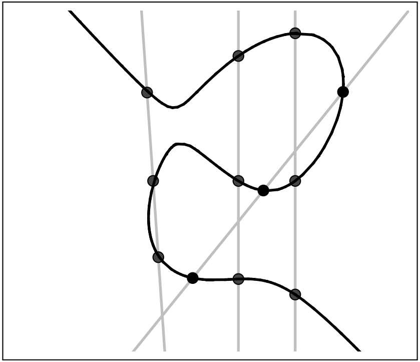
D_4 is obtained from a $(3, 3)$-grid by adding a special set of 3 collinear points.

The proof of Theorem 2 comes from adding (and deleting) collinear sets of points from special grids.

D_4 is a half grid

Definition: A non-grid (a, b)-geproci set Z is a *half grid* if $\overline{Z}$ is the intersection of two curves, exactly one of which can always be taken to be a union of lines.

Example: D_4 is a half grid.



$Z = F_4$ is similar and also motivated our results

$Z = F_4$ is the 24 intersection points of the $\binom{8}{2} = 28$ lines through pairs of vertices of a cube. It contains D_4.

It is $(4,6)$-geproci: it is contained in 6 skew lines (so $\overline{Z}$ lies on a sextic) but in no quadric (so it is a half-grid).

It contains a $(4,4)$-grid (the image of which gives a pencil of quadrics, a member of which is the quartic which contains $\overline{Z}$).

The remaining 8 points are a $(2,4)$-grid (so adding two sets of 4 collinear points to the $(4,4)$-grid gives F_4).

Here is a 3D visualization of F_4.

Open Problems

(a) We know only a few examples of nontrivial geproci non-half grids:

- A 60 point set coming from the H_4 root system (Fraś-Zięba).
- A 40 point $(5, 8)$-geproci set originally constructed by Penrose, who applied it to quantum mechanics (QM).
- A 120 point $(10, 12)$-geproci set also related to QM.

Open Problem 4: Are there only finitely many nontrivial geproci non-half grids?

(b) Every nontrivial geproci set in $\mathbb{P}^3$ that we know of has multiple subsets of at least 3 collinear points.

Open Problem 5: Can a nontrivial geproci set be linearly general?

(c) The 40 point Penrose set is Gorenstein.

Open Problem 6: Are there other finite Gorenstein geproci sets?

More Open Problems

(a) There are, up to projective equivalence, uncountably many grids.

Open Problem 7: Up to projective equivalence, is there any (a, b) with infinitely many nontrivial (a, b)-geproci sets?

(b) We know no example of a geproci set in $\mathbb{P}^n$ for $n > 3$.

Open Problem 8: Do geproci sets exist in $\mathbb{P}^n$, $n > 3$?

(c) We can define a geproci variety as any variety whose general projection is a complete intersection. A cone with a general vertex over a finite geproci set is a geproci curve, the cone over that is a geproci surface, etc. These geproci varieties all have codimension 3.

Open Problem 9: Are there other kinds of geproci varieties? Are there any with codimension greater than 3?

Terao type problems

I.e.: to what extent do the combinatorics of a finite set of points $Z \subset \mathbb{P}^3$ determine its geproci-ness?

Open Problem: Do the collinearities of a finite set of points $Z \subset \mathbb{P}^3$ determine whether or not Z is geproci?

Open Problem: Do the collinearities of geproci sets $Z_1, Z_2 \subset \mathbb{P}^3$ determine whether or not Z_1 and Z_2 are projectively equivalent?

Thanks for your attention!

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