Combinatorics, Geometry and Algebra of Skew lines in $\mathbb{P}^3$

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Organizing Committee: Nancy Abdallah, Nasrin Altafi, Pedro Macias Marques, Chris McDaniel, Juan Migliore, Rosa Maria Miró-Roig, Justyna Szpond, Tomasz Szemberg.

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Slides available eventually at my website (green text is clickable):
https://unlblh.github.io/BrianHarbourne/

But first: the impact of the Lefschetz meetings, 1:

(Thanks to Nancy Abdallah's web page for several of the links.)

Hawaii, 2012; Göttingen, 2015; Mitag-Leffler, 2017.

Banff, 2016: Jean Vallés reported on work with Roberta DiGennaro and Giovanna Ilardi. This work motived the definition of unexpectedness (by David Cook II, Me, Juan Migliore and Uwe Nagel), reported on by Uwe.

Key fact: Unexpectedness gives failures of WLP.

Levico, 2018: Justyna Szpond and I reported on unexpectedness, one of the working groups was on unexpectedness (Alessandra Bernardi, Luca Chiantini, Graham Denham, Giuseppe Favacchio, me, Juan, Tomasz Szemberg and Justyna) and during this group we discovered nontrivial geproci sets, published by Luca and Juan.

Luminy, 2019: More reports on unexpectedness (by Juan and me)

But first: the impact of the Lefschetz meetings, 2:

Oberwolfach, 2020: The POLITUS team (Luca, Łucja Farnik, Giuseppe, me, Juan, Tomasz and Justyna) grew out of this meeting which led to the standard construction (of geproci sets on skew lines)

Cortona, 2022: Tomasz and I reported on geproci sets. POLITUS contributed a short paper to the proceedings volume, studying structure of geproci sets on skew lines, which led to a longer August 2023 paper on skew lines, their combinatorics and connections to geproci sets.

Toronto, 2023: Reports on geproci sets (Łucja) and Weddle locus (Giuseppe). A team (POLITUS+2=POLITUS + Pietro DePoi and Giovanna) worked on geproci sets on skew lines with 2 transversals which led to a December 2023 paper.

Question (POLITUS+2): Do geproci sets on skew lines all come from the standard construction?

But first: the impact of the Lefschetz meetings, 3:

Krakow, 2024: There were reports on Weddle locus (Luca), grids (Giuseppe) and geproci sets on skew lines (me).

Nordfjordeid, 2025: Reports scheduled on geproci sets (me), skew lines (Jake Kettinger) and Fermat configurations (Giovanna).

Also, POLITUS has revised the August 2023 longer paper on skew lines, their combinatorics and connections to geproci sets; it has new results, and answers the POLITUS+2 Question in two ways: Yes for sets of 4 skew lines, but No in general.

What else does the revised much longer POLITUS paper do?

Three perspectives

Combinatorial: the matroid of skew lines

Geometric: geproci sets of points

Algebraic: failures of WLP

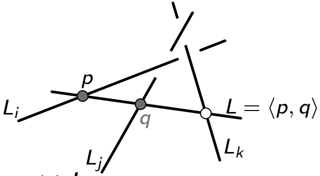
Are these three perspectives equivalent?

Combinatorics (skew lines) = Geometry (geproci)

Given $\mathcal{L} = \{L_1, \ldots, L_s\}$, $s \geq 3$ skew lines in $\mathbb{P}^3$ (over any field).

Given $p \in L_i, q \in L_j, i < j$: Say $p \sim q$ iff

$$\langle p, q \rangle \cap L_k \neq \varnothing, \ k \neq i, j.$$



This generates an equivalence relation $\cong_{\mathcal{L}}$ on $\cup_i L_i$.
(Denote the equivalence class of a point p by $[p]$.)

Say Z has $(*)$ if $Z \subset \cup_i L_i$ satisfies $|Z \cap L_i| = r$ for all i where $r = |Z|/s$. Say a Z with $(*)$ is $(r, \mathcal{L})$-geproci if its projection $\overline{Z}$ from a general point to a plane is a complete intersection of type (r, s) with the image of $\mathcal{L}$ giving the generator of degree s.

Combinatorics = Geometry Theorem (POLITUS): Given Z with $(*)$. Then Z is $(r, \mathcal{L})$-geproci iff Z is a union of finite $\cong_{\mathcal{L}}$ equivalence classes.

Question: Combinatorics = Geometry $\stackrel{?}{=}$ Algebra (WLP)

Given finite set $Z = \{q_1, \ldots, q_t\} \subset \mathbb{P}^3$.

Say Z fails WLP in degree d if

$$\mu_L : [R/(\hat{q}_1{}^d, \ldots, \hat{q}_s{}^d)]_{d-1} \to [R/(\hat{q}_1{}^d, \ldots, \hat{q}_s{}^d)]_d$$

fails to have maximal rank, where L is a general linear form and $\hat{q}_i$ is the linear form dual to q_i.

Conjecture: Say Z has $(*)$ for a set of $s \geq 3$ skew lines $\mathcal{L}$. Then Z is $(r, \mathcal{L})$-geproci iff Z fails WLP in degrees r and s.

Note: this conjecture would have followed in many cases if the answer to the POLITUS+2 Question were Yes.

New Results on the Combinatorics of Skew Lines.

Given $\mathcal{L} = \{L_1, \ldots, L_s\}$, $s \geq 3$ skew lines in $\mathbb{P}^3$ (over any field).

Let $p \in L_i$. There is a group $G_{\mathcal{L}} \subset \mathrm{Aut}(\mathbb{P}^1)$ such that $[p] \cap L_i$ is the $G_{\mathcal{L}}$-orbit of p.

Theorem: (POLITUS)

• $G_{\mathcal{L}}$ is abelian iff there are two or more lines (counted with multiplicity) meeting each line in $\mathcal{L}$.

• If $G_{\mathcal{L}}$ is infinite then all but at most two orbits are infinite, and the finite orbits are singletons. (When $G_{\mathcal{L}}$ is abelian, it acts by scale transformations or translations.)

Open Problem: Classify the finite groups that arise as $G_{\mathcal{L}}$.

Thanks for your attention!

BIRS Focused Research Group
Kelowna Research Station, BC, Canada
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