Asymptotic Invariants of Ideals of Points

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Overview

Waldschmidt's asymptotic invariant $\gamma(I)$.

Historical Motivation for $\gamma(I)$: the Schwarz Lemma.

Bounds on $\gamma(I)$.

Relation to recent work on the containment problem:

- Ein-Lazarsfeld-Smith,
- Hochster-Huneke,
- Bocci-H___

Conjectures and Questions.

Overview (cont.)

- Waldschmidt's asymptotic invariant γ arises in:
  - Number Theory
  - Complex Variables
  - Algebraic Geometry (as a Seshadri constant)
  - Commutative Algebra

- Computing γ is difficult; it is an open problem in general.

- Bounds on γ are useful and are related to the containment problem of when ordinary powers of ideals contain symbolic powers.

- Only a few complete solutions to the containment problem are known (such as for complete intersections, or for the ideal of up to 9 generic points in $\mathbf{P}^2$, or for the ideal I of a finite set of points in $\mathbf{P}^N$ when $\alpha(I) = \mathrm{reg}(I)$).

Notation and Definitions

- $k[\mathbf{P}^N] = k[x_0, \ldots, x_N] = R$ polynomial ring over field k (often $\mathbb{C}$).
- $S = \{p_1, \ldots, p_s\} \subset \mathbf{P}^N$ distinct points.
- $I(p_i) \subset R$ the ideal generated by all forms vanishing at p_i.
- $I(S) = I(p_1) \cap \cdots \cap I(p_s) \subset R$ is a homogeneous ideal.
- mth *symbolic power* of $J = I(p_1)^{m_1} \cap \cdots \cap I(p_s)^{m_s}$:

$$J^{(m)} = I(p_1)^{mm_1} \cap \cdots \cap I(p_s)^{mm_s} \subset R.$$

The Waldschmidt Invariant

- Let $0 \neq J \subseteq R$ be an ideal:

$$\alpha(J) = \text{ least } t \text{ such that } J \text{ has an element of degree } t.$$

Lemma (Waldschmidt 1975)

The limit $\gamma(I(S)) = \lim_{m\to\infty} \dfrac{\alpha(I(S)^{(m)})}{m}$ *exists.*

Intuition: $m\gamma(I(S))$ is the approximate minimum degree of a form vanishing on S to order m for $m \gg 0$.

Conjectures and Questions

Properties of powers, symbolic powers, α and γ

Given $S = \{p_1, \dots, p_s\} \subset \mathbf{P}^N$, let $I = I(S)$:

- $I = I^{(1)}$
- $I^m \subseteq I^{(m)}$ for all $m \geq 1$
- α is linear in ordinary powers: $\alpha(I^m) = m\alpha(I)$
- α is sublinear in symbolic powers: $\alpha(I^{(i+j)}) \leq \alpha(I^{(i)}) + \alpha(I^{(j)})$
- γ is linear in symbolic powers: $\gamma(I^{(m)}) = m\gamma(I)$

Sublinearity of $\alpha(I^{(m)})$ and failure of $I^m = I^{(m)}$.

Given $S = \{p_1, \cdots, p_s\} \subset \mathbf{P}^N$, then $I(S)^m \subseteq I(S)^{(m)}$.

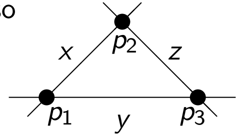
Proof: $I(S)^m \subseteq I(p_j)^m \Rightarrow I(S)^m \subseteq \cap_j I(p_j)^m = I(S)^{(m)}$.

Example: $I(S)^m = I(S)^{(m)}$ can fail. Take 3 general points in $\mathbf{P}^2$:

$I = I(p_1, p_2, p_3)$ is generated in degree 2 so

$\alpha(I^2) = 2\alpha(I) = 4$ hence $xyz \notin I^2$,

but $xyz \in I^{(2)}$, hence $I^2 \subsetneq I^{(2)}$



Note sublinearity: $3 = \alpha(I^{(2)}) < 2\alpha(I) = 4$.

Conjectures and Questions

Open Problem: Compute $\gamma(I(S))$.

Computing $\gamma(I(S))$ is hard. Two Open Problems:

Problem: Given any finite set $S \subset \mathbf{P}^N$, compute $\gamma(I(S))$.

Problem: $s > 9$ generic points $S \subset \mathbf{P}^2$, $\sqrt{s} \notin \mathbb{Z}$: Show $\gamma(I(S)) = \sqrt{s}$.

Some sample results on $\gamma(I(S))$:

- $s \leq 9$ generic points $S \subset \mathbf{P}^2$:

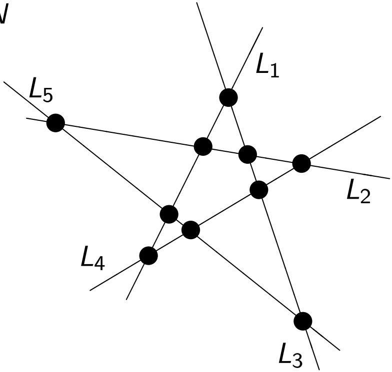
s	1	2	3	4	5	6	7	8	9
$\gamma(I(S))$	1	1	$\frac{3}{2}$	2	2	$\frac{12}{5}$	$\frac{21}{8}$	$\frac{48}{17}$	3

- Seshadri constant $\varepsilon(S) = \left(\frac{\gamma(I(S))}{s}\right)^{\frac{1}{N-1}}$ for s generic points $S \subset \mathbf{P}^N$

Examples continued...

- $S =$ the set of points of all N-wise intersections of $s \geq N$ general hyperplanes in $\mathbf{P}^N$: $\gamma(I(S)) = \dfrac{s}{N}$

Diagram: $N = 2$, $s = 5$ ($S =$ the 10 points of pair-wise intersections of $s = 5$ lines in $\mathbf{P}^2$): $\gamma(I(S)) = \dfrac{5}{2}$



Some transcendence theory

Theorem (Schneider 1941)

$a, b \in \mathbb{Q} \setminus \mathbb{Z} \ \Rightarrow\ \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ *is transcendental over* $\mathbb{Q}$.

Example

$$\text{Recall: } \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \text{ for } z \notin \{-1, -2, -3, \ldots\}.$$

$$\text{Therefore } \frac{\Gamma(\frac{1}{2})^2}{\Gamma(1)} = \frac{\frac{\pi}{\sin(\frac{\pi}{2})}}{1!} = \pi \text{ is transcendental over } \mathbb{Q}.$$

Proof of Theorem: Uses multi-variable multi-zero Schwarz Lemma. □

What is the Schwarz Lemma?

Lemma (Classical one variable one zero Schwarz Lemma)

Let $F(z)$ be complex analytic with 0 of order at least m at $z = 0$. Given $r > 0$, there exists $C > 0$ such that for $|z| < r$ we have:

$$|F(z)| \leq C|z|^m.$$

Proof: Immediate application of Maximum Modulus Principle applied to the function

$$G(z) = \frac{F(z)}{z^m}$$

which is analytic since F has a zero of order at least m at $z = 0$.

What is the Schwarz Lemma? DETAILS

Given $F : \mathbb{C} \to \mathbb{C}$ and real $r > 0$, let $|F(z)|_r = \max_{|z|=r} |F(z)|$.

Lemma (Classical one variable one zero Schwarz Lemma)

Given complex analytic function $F(z)$ with 0 of order at least m at $z = 0$. Then for $0 \leq |z_0| < r$ and $C = \frac{|F(z)|_r}{r^m}$ we have:

$$|F(z_0)| \leq |F(z)|_r \left(\frac{|z_0|}{r}\right)^m = C|z_0|^m.$$

Proof: $G(z) = \dfrac{F(z)}{z^m}$ is analytic, so apply Max Modulus Principle:

$$|F(z_0)| = |G(z_0)| \cdot |z_0|^m \leq |G(z)|_r \cdot |z_0|^m = |F(z)|_r \left(\frac{|z_0|}{r}\right)^m \quad \square$$

Example of applying Schwarz Lemma in number theory

Claim: $\sqrt{2} \notin \mathbb{Q}$

Exercise: There are integers $x, y > 0$ with $x^2 - 2y^2 = 1$ such that:

- y is arbitrarily large
- $|\frac{x}{y} - \sqrt{2}| < \frac{1}{y^2}$
- $\frac{1}{sy} < |(\frac{x}{y})^2 - (\frac{r}{s})^2|$ for any fixed rational $\frac{r}{s}$ and all $y \gg 0$

Proof of claim: Assume $\sqrt{2} = \frac{r}{s} \in \mathbb{Q}$. Apply Schwarz Lemma to: $F(z) = (z + \sqrt{2})^2 - 2$ with $m = 1$. For $y \gg 0$, $z = \frac{x}{y} - \sqrt{2}$ gives

$$\frac{1}{sy} < \left|\left(\frac{x}{y}\right)^2 - \left(\frac{r}{s}\right)^2\right| = \left|\left(\frac{x}{y}\right)^2 - 2\right| \leq C\left|\frac{x}{y} - \sqrt{2}\right| < C\frac{1}{y^2}$$

$$\text{so } 1 < C\frac{s}{y} \text{ for } y \gg 0\text{: contradiction.} \quad \square$$

The Schwarz Lemma and $\gamma(I(S))$

Restate the Schwarz Lemma for $0 < |z| = r_1 < r$:

$$\log |F(z)|_{r_1} \leq m \log(r_1) + \log C$$

where $|F(z)|_{r_1} = \max_{|z|=r_1} |F(z)|$.

Waldschmidt's multi-variable multi-zero version (1976):

Given entire $F : \mathbb{C}^N \to \mathbb{C}$ vanishing to order m or more at each point of a finite set $S \subset \mathbb{C}^N$, then for $0 \ll r_0 \leq r_1 < r$, there is a constant $C > 0$ (of a very particular form) such that

$$\log |F(\underline{z})|_{r_1} \leq m\,\gamma(I(S)) \log r_1 + \log C$$

where $|F(\underline{z})|_{r_1} = \max_{|\underline{z}|=r_1} |F(\underline{z})|$.

The Schwarz Lemma and $\gamma(I(S))$ DETAILS

Restate the Schwarz Lemma for $0 < |z| = r_1 < r$:

$$\log|F(z)|_{r_1} \leq m\log(r_1) + \log C$$

where $|F(z)|_{r_1} = \max_{|z|=r_1} |F(z)|$.

Waldschmidt's multi-variable multi-zero version (1976): For every finite subset $S \subset \mathbb{C}^N$ and every $0 < \epsilon < 1$, there exists a constant $r_0 = r_0(S, \epsilon)$ such that for every $m > 0$ and every entire function $F : \mathbb{C}^N \to \mathbb{C}$ vanishing to order at least m on S, we have

$$\log|F(\underline{z})|_{r_1} \leq \log|F(\underline{z})|_r + m(1-\epsilon)(\gamma(S)-\epsilon)\log(\frac{6Nr_1}{\epsilon r})$$

for all $r_0 \leq r_1 < r$, where $|F(\underline{z})|_{r_1} = \max_{|\underline{z}|=r_1} |F(\underline{z})|$.

Waldschmidt's and Skoda's Bounds on $\gamma(I(S))$ (1976)

Waldschmidt: Given $S = \{p_1, \cdots, p_s\} \subset \mathbf{P}^N$.

Let $I = I(S)$.

$$\cancel{2+}\ \frac{\alpha(I)}{N} \le \overbrace{\lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}}^{\gamma(I)} \le \frac{\alpha(I^{(m)})}{m} \le \alpha(I)$$

Skoda

(*)
hard

pretty easy
(α is sublinear)

easy

Proof of (*): Uses complex analytic methods.

Uses refinements of results of Bombieri on plurisubharmonic functions.

Alternate Proof of (*)

Theorem (Ein-Lazarsfeld-Smith 2001 / Hochster-Huneke 2003)

Let $I \subseteq k[\mathbf{P}^N]$ be a homogeneous ideal and $m > 0$. Then

$$I^{(mN)} \subseteq I^m.$$

Proof: Find an ideal J such that $I^{(mN)} \subseteq J \subseteq I^m$.
ELS: uses multiplier ideals
HH: uses Frobenius powers and tight closure $\square$

Alternate Proof of (*) $\frac{\alpha(I)}{N} \leq \gamma(I)$: $I^{(mN)} \subseteq I^m \Rightarrow$

$$\frac{\alpha(I)}{N} = \frac{m\alpha(I)}{mN} = \frac{\alpha(I^m)}{mN} \leq \frac{\alpha(I^{(mN)})}{mN} \xrightarrow[m\to\infty]{} \gamma(I) \qquad \square$$

γ and the resurgence ρ

Given finite subset $S \subset \mathbf{P}^N$ and $I = I(S)$.

The Containment Problem: Find all m and r with $I^{(m)} \subseteq I^r$.

Definition (the resurgence: Bocci-H___): $\rho(I) = \sup\{\frac{m}{r} : I^{(m)} \not\subseteq I^r\}$.

Theorem

(a) *If* $\frac{m}{r} > \rho(I)$, *then* $I^{(m)} \subseteq I^r$.

(b) $\rho(I) \leq N$

(c) *(Bocci-H___ JAG 2009)* $\dfrac{\alpha(I)}{\gamma(I)} \leq \rho(I) \leq \dfrac{\mathrm{reg}(I)}{\gamma(I)}$

(d) *(Bocci-H___)* $\alpha(I) = \mathrm{reg}(I) \Rightarrow I^{(m)} \subseteq I^r$ *iff* $r\alpha(I) \leq \alpha(I^{(m)})$

Proof of the theorem

(a) If $\frac{m}{r} > \rho(I)$, then $I^{(m)} \subseteq I^r$: Immediate from definition of $\rho(I)$.

(b) $\rho(I) \leq N$: Immediate from ELS-HH result and definition of $\rho(I)$.

(c) Proof of the lower bound $\frac{\alpha(I)}{\gamma(I)} \leq \rho(I)$:

It is enough to show $\dfrac{mt}{rt} < \dfrac{\alpha(I)}{\gamma(I)} \Rightarrow I^{(mt)} \not\subseteq I^{rt}$ for $t \gg 0$.

- $\dfrac{mt}{rt} < \dfrac{\alpha(I)}{\gamma(I)} \Rightarrow mt \lim\limits_{s\to\infty} \dfrac{\alpha(I^{(ms)})}{ms} = mt\gamma(I) < rt\alpha(I) = \alpha(I^{rt})$

$$\Rightarrow \alpha(I^{(mt)}) = mt\frac{\alpha(I^{(mt)})}{mt} < \alpha(I^{rt}) \text{ for } t \gg 0$$

- But $\alpha(I^{(mt)}) < \alpha(I^{rt}) \Rightarrow I^{(mt)} \not\subseteq I^{rt}$.

(c) Proof of the upper bound $\rho(I) \leq \frac{\mathrm{reg}(I)}{\gamma(I)}$

It is enough to show $\frac{\mathrm{reg}(I)}{\gamma(I)} \leq \frac{m}{r} \Rightarrow I^{(m)} \subseteq I^r$. Assume $\frac{\mathrm{reg}(I)}{\gamma(I)} \leq \frac{m}{r}$.

- $1 \leq \frac{\mathrm{reg}(I)}{\gamma(I)}$ so $r \leq m$ so $I^{(m)} \subseteq I^{(r)}$ hence $(I^{(m)})_t \subseteq (I^{(r)})_t$ for $t \geq 0$
- $t \geq r\mathrm{reg}(I) \Rightarrow (I^{(r)})_t = (I^r)_t$ (Geramita-Gimigliano-Pitteloud '95)
- $t < r\mathrm{reg}(I) \leq m\gamma(I) \leq \alpha(I^{(m)}) \Rightarrow (I^{(m)})_t = 0 \subseteq (I^r)_t$.
- Thus $(I^{(m)})_t \subseteq (I^r)_t$ for all $t \geq 0$ so $I^{(m)} \subseteq I^r$.

Proof of (d): Similar arguments.

Are W-S & ELS/HH Optimal?

Given: finite subset $S \subset \mathbf{P}^N$:

W-S inequality: $\frac{\alpha(I(S))}{\gamma(I(S))} \leq N$.

Restatement of ELS-HH result: $\rho(I(S)) \leq N$.

Note: Restated ELS-HH result $\Rightarrow$ W-S inequality, since

$$\frac{\alpha(I(S))}{\gamma(I(S))} \leq \rho(I(S)).$$

Question: Are the results of W-S & ELS/HH optimal?

Can the upper bound N be improved?

Are W-S & ELS/HH Optimal?

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Can the upper bound N be improved?

Theorem (Bocci-H___): W-S is optimal since

$$\sup_{|S|<\infty}\left\{\frac{\alpha(I(S))}{\gamma(I(S))}\right\} = N.$$

Corollary: Therefore ELS-HH is optimal, since

$$\frac{\alpha(I(S))}{\gamma(I(S))} \leq \rho(I(S)) \ \Rightarrow \ N \leq \sup_{|S|<\infty}\{\rho(I(S))\}.$$

Nonetheless, can W-S or ELS-HH be improved?

Fact 1 (Chudnovsky 1980): If $S \subset \mathbf{P}^2$ is finite, then

$$\frac{\alpha(I(S))+1}{2} \leq \gamma(I(S))$$

is a sharp lower bound.

Proof: Geometric. $\square$

Conjecture 1 (Chudnovsky 1980): If $S \subset \mathbf{P}^N$ is finite, then

$$\frac{\alpha(I(S))+N-1}{N} \leq \gamma(I(S)).$$

Note: If Conjecture 1 is true, then it is sharp.

Possible upper bound on ρ?

Conjecture (H___ 2009)

Let $S \subset \mathbf{P}^2$ be finite. Then $\rho(I(S)) \leq 2\dfrac{\alpha(I(S))}{\alpha(I(S))+1}$.

The Conjecture implies Chudnovsky's bound:

$$\frac{\alpha(I(S))}{\gamma(I(S))} \leq \rho(I(S)) \leq 2\frac{\alpha(I(S))}{\alpha(I(S))+1} \Rightarrow \frac{\alpha(I(S))+1}{2} \leq \gamma(I(S)).$$

Possible upper bounds on ρ

Let $S \subset \mathbf{P}^N$ be finite. Then

$$\rho(I(S)) \leq N\frac{\alpha(I(S))}{\alpha(I(S)) + N - 1}$$

fails for all $N > 2$ due to examples with S in a hyperplane.

Perhaps we can avoid the counterexamples:

Is $\rho(I(S)) \leq \max\left(N - 1 + \frac{2}{N(N+1)}, N\frac{\alpha(I(S))}{\alpha(I(S)) + N - 1}\right)$?

Additional possibilities

Examples of Takagi & Yoshida suggest $I(S)^{(rN-1)} \subseteq I(S)^r$.

Question (Huneke 2003)

For $I = I(p_1, \ldots, p_s) \subset k[\mathbf{P}^2]$, is it true that $I^{(3)} \subseteq I^2$?

Answer: Yes, in characteristic 2! (Open in general.)

Examples suggest even more:

Conjecture (H___ 2008)

Let $S \subset \mathbf{P}^N$ be finite. Then $I(S)^{(rN-(N-1))} \subseteq I(S)^r$ for all $r > 0$.

Supporting Evidence

- $I(S)^{(rN-(N-1))} \subseteq I(S)^r$ and $\rho(I(S)) \leq N\dfrac{\alpha(I(S))}{\alpha(I(S))+N-1}$

both hold for generic sets S of points in $\mathbf{P}^2$, and also for generic sets S of points in $\mathbf{P}^N$ when the number of points is sufficiently large.

Also, let $S \subset {\mathbf{P}^N}_k$ be finite. Then $I(S)^{(rN-(N-1))} \subseteq I(S)^r$ holds:

- when $r = q^i$ and $q = \mathrm{char}(k) > 0$; and
- in all characteristics for all r for monomial ideals $I(S)$.

Proof of $I(S)^{(rN-(N-1))} \subseteq I(S)^r$ for monomial ideals:

Let $I = I(S)$ be monomial. Then $I(p)$ is monomial for each $p \in S$.

Let $I^{[q]}$ be the ideal generated by all qth powers of monomials in I.

Fact (1): P prime and monomial, Q monomial and P-primary $\Rightarrow$ $Q = \cap_i J_i$ for finitely many N-generated primary monomial ideals J_i.

Fact (2): J monomial, N-generated, $m \geq rN - (N-1) \Rightarrow J^m \subseteq J^{[r]}$.

Fact (3): J_i monomial $\Rightarrow (\cap_i J_i)^{[r]} = \cap_i J_i^{[r]}$.

Let $m = rN - (N-1)$. Then:

$$I^{(m)} = \cap_{p\in S} I(p)^m \underset{(1)}{=} \cap_{p\in S} (\cap_i J_i(p))^m \subseteq \cap_{p,i}(J(p)_i)^m$$

$$\underset{(2)}{\subseteq} \cap_{p,i} (J(p)_i)^{[r]} \underset{(3)}{=} (\cap_{p,i} J(p)_i)^{[r]} \subseteq (\cap_{p,i} J(p)_i)^r = I^r$$

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- Only a few complete solutions to the containment problem are known (such as for complete intersections, or when $\alpha(I) = \mathrm{reg}(I)$ for the ideal I of a finite set of points in $\mathbf{P}^N$, or for the ideal of up to 9 generic points in $\mathbf{P}^2$).