Algebraic Geometric Concepts Motivated by Inverse Scattering

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Slides will be available at my website (green text is clickable):
https://www.math.unl.edu/~bharbourne1/

Main reference: arXiv:2209.04820

This talk is on joint work of the Geproci Team (GT):

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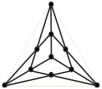
POLITUS: POLand, ITaly and the US

An international collaboration of 7 researchers whose logo is a stylized D_4 configuration:



Connections

Our work has connections to:

Algebraic Geometry

Commutative Algebra

Combinatorics

Representation Theory

Quantum Physics

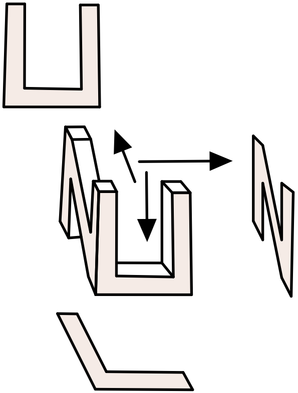
It can also be considered in a larger not-only-mathematical context!

Abstract

Studying inverse scattering problems has led to remarkable advances in scientific knowledge. Here we propose carrying this idea over to classification problems in algebraic geometry.

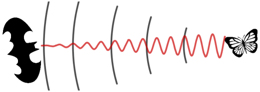
Inverse scattering Problems (ISP): try to discern structure from projected or reflected data.

Idea: classify structures algebro-geometrically based on properties of projected images.

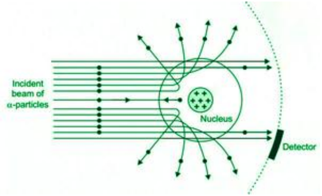


Some examples of ISP

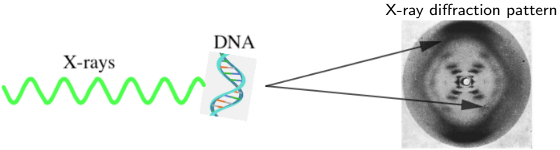
Echolocation (biology):



Rutherford scattering (physics; led to Bohr model of atom):



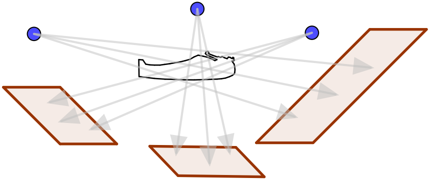
X-ray crystallography (chem/bio; led to DNA double helix model):



Rosalind Franklin
"Photo 51"

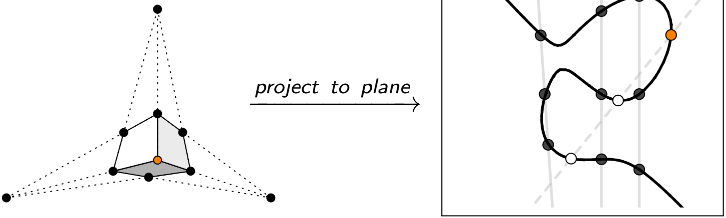
More examples

Tomography (medicine):



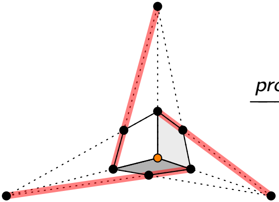
GePro-$\mathcal{P}$ (math): Pick a property $\mathcal{P}$ and classify finite point sets in $\mathbb{P}^n$ whose **Ge**neral **Pro**jections to a hyperplane satisfy $\mathcal{P}$.

Here are 12 points (10 visible) in space whose projections from general points to a plane are complete intersections (so $\mathcal{P}$ is "being a CI"). These 12 points (known as D_4) are "geproci."

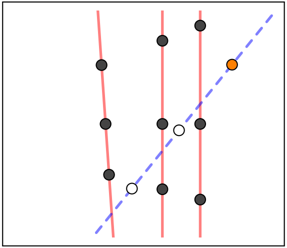


Why D_4 is geproci

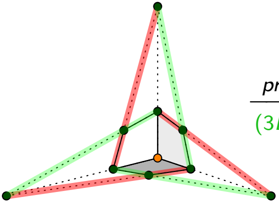
The quartic comes from lines through collinear points:



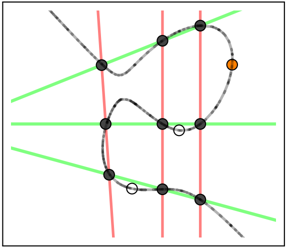
project to plane $\longrightarrow$



The cubic is one in a pencil of cubics:



project to plane $\longrightarrow$
(3*D visualization*)



Gepro-$\mathcal{P}$

General Problem: Given a property $\mathcal{P}$ of finite point sets $\overline{Z} \subset \mathbb{P}^{n-1}$, classify all finite $Z \subset \mathbb{P}^n$ such that $\overline{Z} \subset H \cong \mathbb{P}^{n-1}$ has property $\mathcal{P}$ (where $\overline{Z}$ is the image of Z under projection $\mathbb{P}^n \dashrightarrow H$ from a general point P to a hyperplane $H \subset \mathbb{P}^n$).

Example 1: Say $\mathcal{P}$ means "$\overline{Z}$ is Gorenstein". Then a set Z of $n+1$ general points in $\mathbb{P}^n$ is gepro-$\mathcal{P}$ since the image $\overline{Z}$ is a set of $n+1$ general points in H, which is Gorenstein.

Open Problem 1: Classify gepro-Gorenstein sets Z.

Every geproci set is also gepro-Gorenstein but not conversely.

Open Problem 2: Classify geproci sets in $\mathbb{P}^n$.

History of geproci

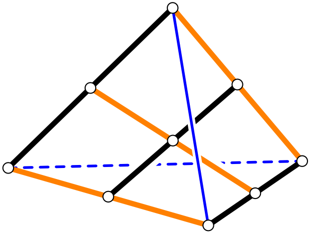
We know no interesting examples of geproci sets in $\mathbb{P}^n$ for $n > 3$.

We say $Z \subset \mathbb{P}^3$ is (a, b)-geproci if $\overline{Z}$ is the intersection of a curve A of degree a with a curve B of degree b, with $a \leq b$.

A geproci set Z in a plane $H \subset \mathbb{P}^3$ is called *degenerate*; it is just the complete intersection of two curves in H.

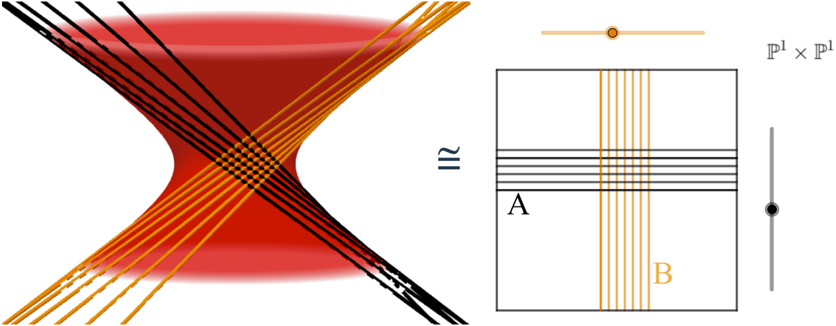
Question 1 (F. Polizzi 2011): Is every geproci set in $\mathbb{P}^3$ degenerate?

Answer (D. Panov, 2011): No!
(a, b)-grids are nondegenerate and geproci. I.e., $2 \leq a \leq b$ with A being a skew black lines and B being b skew orange lines, where each black line intersects each orange line in exactly 1 point. (Here $a = b = 3$.)



We understand grids.

Fact: For an (a, b)-grid with $3 \leq a \leq b$, the grid lines come from the rulings on a smooth quadric.



Fact: A $(2, b)$-grid consists of b points on each of two skew lines (but the grid lines need not all lie on a smooth quadric).

New Question and a partial answer

For simplicity, call a geproci set in $\mathbb{P}^3$ *trivial* if it is either a grid or contained in a plane.

Question 1′ (F. Polizzi 2011): Is every geproci $Z \subset \mathbb{P}^3$ trivial? If not, can such Z be classified up to projective equivalence, at least when $|Z|$ is small?

Answer (2018, Lefschetz Working Group at Levico Terme): Certain finite sets Z given by root systems (such as D_4 and F_4) which have unexpected cones (see Harbourne-Migliore-Nagel-Teitler: arXiv:1805.10626, Michigan Math. J. 2020) turn out to be nontrivial geproci sets.

Theorem (Levico Terme Working Group, 2018): A finite set $Z \subset \mathbb{P}^3$ is (a, b)-geproci if $|Z| = ab$ and it has unexpected cones of degrees $a \leq b$ with no common components.

The 2018 Levico Terme Working Group (LTWG)

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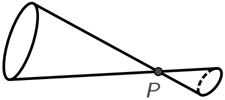
Unexpected cones

Let $p_1, \dots, p_s \in \mathbb{P}^n$ be distinct and let $P \in \mathbb{P}^n$ be general.

Let $Z = \{p_1, \dots, p_s\}$ and $I(Z) \subset k[\mathbb{P}^n] = k[x_0, \dots, x_n]$ its ideal.

The cones of degree t with vertex P: $[I(P)^t]_t$



The cones of degree t with vertex P containing Z: $[I(Z)]_t \cap [I(P)^t]_t$

They're "unexpected" if there are more than expected:

$$\dim\left([I(Z)]_t \cap [I(P)^t]_t\right) > \max\left\{0, \dim[I(Z)]_t - \binom{n+t-1}{n}\right\}.$$

We write "Z satisfies $C(t)$" if Z has unexpected cones of degree t.

A result and an Open Problem

Theorem (Chiantini-Migliore, arXiv:1904.02047, TAMS 2021):
Every (a, b)-grid with $3 \leq a \leq b$ satisfies both $C(a)$ and $C(b)$.

Open Problem 3: Does every nontrivial (a, b)-geproci $Z \subset \mathbb{P}^3$ satisfy both $C(a)$ and $C(b)$?

D_4 played a special role; F_4 was important too!

Theorem (Chiantini-Migliore, arXiv:1904.02047, TAMS 2021): The least $|Z|$ for a nontrivial geproci set is $|Z| = 12$. An example is given by the D_4 configuration of 12 points; it is $(3, 4)$-geproci.

Theorem 1 (GT, 2022): The D_4 configuration is, up to projective equivalence, the only nontrivial (a, b)-geproci set in $\mathbb{P}^3$ with $a \leq 3$.

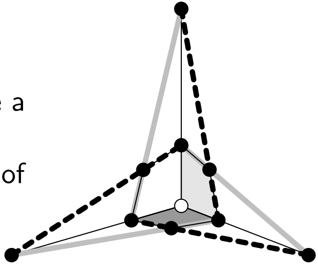
D_4 and F_4 motivated the following theorem:

Theorem 2 (GT, 2022): For each $4 \leq a \leq b$, there is a nontrivial (a, b)-geproci $Z \subset \mathbb{P}^3$.

The proof of Theorem 2 starts with specific (a, a)-grids and adds one (or two) specific set(s) of a collinear points, as exemplified by D_4 (or F_4), followed by deletions of certain collinear subsets.

How D_4 and F_4 motivated Theorem 2

D_4 is at right: the gray and dashed lines give a $(3,3)$-grid. The main diagonal of the cube through the white point is the additional set of 3 collinear points.



F_4 is the 24 intersection points of the $\binom{8}{2} = 28$ lines through pairs of vertices of a cube.

Fact: It has unexpected cones of degrees 4 and 6, and $|F_4| = 24$.

Conclusion (LTWG): F_4 is a $(4,6)$-geproci set.

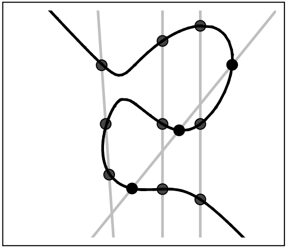
For F_4, $\overline{Z}$ is the intersection of 6 lines with an irreducible quartic.

Here is a 3D view of F_4, its $(4,4)$-subgrid and its unexpected cones.

D_4 and F_4 are half grids

Definition: A nontrivial (a, b)-geproci set Z is a *half grid* if $\overline{Z}$ is the intersection of two curves, exactly one of which can always be taken to be a union of lines.

Example: $Z = D_4$ is not contained in a smooth quadric so it is not a grid. Here $\overline{Z}$ is contained in 4 lines, so it is a half grid.



Example: $Z = F_4$ is not contained in a smooth quadric so it is not a grid. Here $\overline{Z}$ is contained in 6 lines, so it is a half grid.

Open Problems

(a) We know only a few examples of nontrivial geproci non-half grids:
- The 60 point set for the H_4 root system (Wiśniewska-Zięba).
- A 40 point $(5, 8)$-geproci set applied by Penrose to quantum mechanics (QM).
- A 120 point $(10, 12)$-geproci set also related to QM.

Open Problem 4: Are there only finitely many nontrivial geproci non-half grids?

(b) Every nontrivial geproci set in $\mathbb{P}^3$ that we know of has multiple subsets of at least 3 collinear points.

Open Problem 5: Can a nontrivial geproci set be linearly general?

(c) The 40 point Penrose set is Gorenstein.

Open Problem 6: Are there other finite Gorenstein geproci sets?

More Open Problems

(a) There are, up to projective equivalence, uncountably many grids.

Open Problem 7: Up to projective equivalence, is there any (a, b) with infinitely many nontrivial (a, b)-geproci sets?

(b) We know no example of a geproci set in $\mathbb{P}^n$ for $n > 3$.

Open Problem 8: Do geproci sets exist in $\mathbb{P}^n$, $n > 3$?

(c) We can define a geproci variety as any variety whose general projection is a complete intersection. A cone with a general vertex over a finite geproci set is a geproci curve, the cone over that is a geproci surface, etc. These geproci varieties all have codimension 3.

Open Problem 9: Are there other kinds of geproci varieties? Are there any with codimension greater than 3?

Terao type problems

Terao's Conjecture concerns whether a certain property of hyperplane arrangements is a combinatorial property. A geproci set also has combinatorics (e.g., its collinear subsets).

Open Problem 10: If two geproci sets have the same combinatorics, are they projectively equivalent?

Open Problem 11: If a set has the same combinatorics as a geproci set, is it geproci?

Other work.

Jake Kettinger: Exploring geproci in positive characteristics.

Theorem: Let k be a finite field, $q = |k|$, $\overline{k}$ its algebraic closure and let Z be all k-points of $\mathbb{P}^3_{\overline{k}}$. Then Z is a nontrivial $(q+1, q^2+1)$-geproci set.

Frank Zimmitti: Exploring more general definitions of unexpectedness.

Thanks for your attention!

FINIS