New work on unexpected varieties *(and line arrangements)* with a short history and connection to Lefschetz properties

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Workshop on Lefschetz Properties in Algebra, Geometry and Combinatorics II

Centre International de Rencontres Mathématiques

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Slides available at:
http://www.math.unl.edu/~bharbourne1/LuminySlides.pdf

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Set up: Strong Lefschetz Property (SLP)

Note: We work over $\mathbb{C}$ for this whole talk.

Let $R = \mathbb{C}[x_0, \ldots, x_r]$, and let $L \in R$ be a general linear form.

Let $\mathcal{L} = \{L_1, \ldots, L_r\} \subset R$ be linear forms defining distinct hyperplanes in $\mathbb{P}^r$.

Let $I = (L_1^{t_1}, \ldots, L_s^{t_s})$. Assume $A = R/I$ is artinian.

Definition: We say A (or I) *fails SLP* in range k and degree d if

$$A_{d-k} \xrightarrow{\times L^k} A_d$$

does not have maximal rank.

Questions

Can we hope to classify all such failures of SLP?

Probably not in general, but yes in special cases!

Can we always hope to relate all such failures to geometry?

Probably not in general, but yes in special cases!

Aspirational Research Goal: Do both when $\mathcal{L}$ is a supersolvable line arrangement.

Let's look at some already known special cases.

Some history

Recall $I = (L_1^{t_1}, \ldots, L_s^{t_s}) \subset R = \mathbb{C}[x_0, \ldots, x_r]$ with $A = R/I$ artinian.

• HMNW (Harima, Migliore, Nagel, Watanabe: J. Alg., 2003):

Theorem: If $R = \mathbb{C}[x_0, x_1, x_2]$ and J is a complete intersection (CI) and R/J artinian, then R/J satisfies SLP in range 1 (i.e., there are no failures in 3 variables in range 1 when J is a CI).

• SS (Schenck, Seceleanu: PAMS, 2009):

Theorem: If $R = \mathbb{C}[x_0, x_1, x_2]$, then A satisfies SLP in range 1 (i.e., there are no failures in 3 variables in range 1 ever).

• MMN (Migliore, Miró-Roig, Nagel: Alg. Number Thry, 2012; $s = 4$); AA (Almeida, Andrade: Forum Math.), MN (Migliore, Nagel: J. Comm. Alg.; $s \geq 4$, both 2017):

Theorem: If $R = \mathbb{C}[x_0, x_1, x_2]$, L_i general and $I = (L_1^{t_1}, \ldots, L_s^{t_s})$, then $A = R/I$ satisfies SLP in range 2 (i.e., there are no failures in 3 variables in range 2).

More history and a little geometry

Failures can occur and sometimes they are related to geometry:

• HSS (BH, Schenck, Seceleanu: JLMS, 2011), MMN (ANT, 2012): for general L_i and $I = (L_1^t, \dots, L_s^t) \subset R = \mathbb{C}[x_0, \dots, x_r]$, found failures and related them by inverse systems to geometry of fat points.

Let $\mathcal{P} = \{p_1, \dots, p_s\}$ be distinct points in $\mathbb{P}^r$, $\mathcal{L} = \{L_1, \dots, L_s\}$ linear forms defining distinct hyperplanes in $\mathbb{P}^r$.

Write $\mathcal{L}_\mathcal{P}$ for the linear forms dual to the points in $\mathcal{P}$, and $\mathcal{P}_\mathcal{L}$ for the points dual to the linear foms in $\mathcal{L}$.

• FV (Faenzi, Vallès, JLMS, 2014): for $r = 2$, relates existence of plane curves of degree d vanishing on $\mathcal{P} = \{p_1, \dots, p_s\}$ with general point of multiplicity $m = d - 1$ to splitting type of the syzygy bundle $\mathcal{S}_{\mathcal{L}_\mathcal{P}}$ of the Jacobian ideal of $I = (L_1 \cdots L_s)$ for $\mathcal{L}_\mathcal{P} = \{L_1, \cdots, L_s\}$.

More history and a little geometry (continued)

Failures can occur and sometimes they are related to geometry:

• DIV (Di Gennaro, Ilardi, Vallès, JLMS 2014): relates existence of failures of SLP in range 2 and degree $t-2$ for $I = (L_1^t, \dots, L_s^t) \subset R = \mathbb{C}[x_0, x_1, x_2]$, to splitting type of $\mathcal{S}_{\mathcal{L}}$.

• CHMN (Cook, BH, Migliore, Nagel, Compositio 2018): using results of and inspired by FV and DIV, relates failures of SLP to existence of *unexpected curves*; characterizes but does not classify unexpected curves.

There is all sorts of interesting geometry related to unexpected curves. E.g.:

• BMSS (Bauer, Malara, Szemberg, Szpond, Manus. Math. 2018), HMNT (BH, Migliore, Nagel, Teitler, Mich. J. Math. to appear): BMSS identified a geometric duality satisfied by unexpected curves, further studied by HMNT, but it's not yet clear how or if this duality is reflected in Lefschetz properties.

Set up: Recall what *unexpected curves* are

Let $\mathcal{P} = \{p_1, \dots, p_r\}, p_i \subset \mathbb{P}^2$, with a general point $p \in \mathbb{P}^2$.

Let $I(\mathcal{P}) = I(p_1, \dots, p_r)$.

One expects vanishing at p to order m to impose $\binom{m+1}{2}$ conditions.

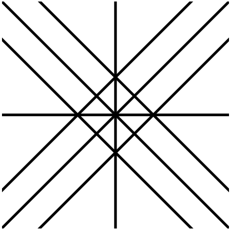
Definition: We say $I(\mathcal{P})$ has an *unexpected* curve of degree d if

$$\dim[I(\mathcal{P}) \cap (I(p)^{d-1})]_d > \max\left(0, \dim[I(\mathcal{P})]_d - \binom{m+1}{2}\right),$$

where $m = d - 1$. I.e., there are more curves of degree d with a general point of multiplicity $m = d - 1$ and containing $\mathcal{P}$ than expected.

B_3: the example from DIV motivating CHMN

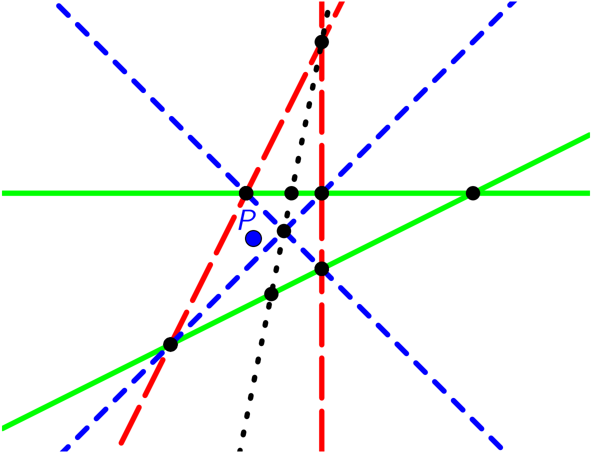
The B_3 arrangement of 9 lines (the line at infinity is not shown):



(The nine lines of $\mathcal{L}_{B_3}$ are dual to the roots of the B_3 root system.)

Constructing the points dual to B_3 geometrically

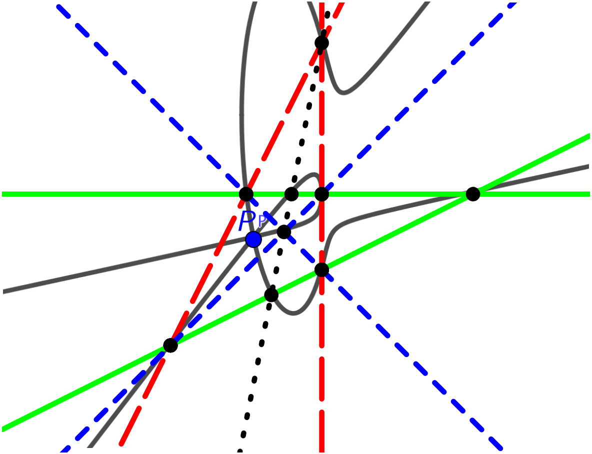
4 general points $\longrightarrow$ 3 singular conics $\longrightarrow$ $4+3+2=9$ points:



The unexpected quartic

The 9 points impose 9 conditions on quartics, so $\dim[I(\mathcal{P}_{\mathcal{L}_{B_3}})]_4 = 6$.

Vanishing at p to order 3 should impose 6 conditions in general, but unexpectedly it imposes only 5. Here is the unexpected quartic C (in black).



BMSS duality

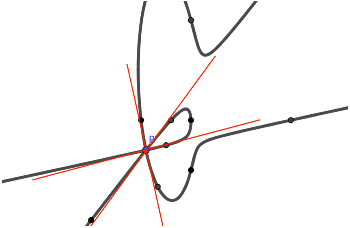
Let $p = (a : b : c)$. Then the form $F(a, b, c, x, y, z)$ defining C is

$$c^3x^3y - c^3xy^3 - b^3x^3z + (3ab^2 - 3ac^2)x^2yz + (-3a^2b + 3bc^2)xy^2z + a^3y^3z + (3a^2c - 3b^2c)xyz^2 + b^3xz^3 - a^3yz^3.$$

Now think of $F(a, b, c, x, y, z)$ as a curve D in the variables a, b, c:

$$= (y^3z - yz^3)a^3 - 3xy^2za^2b + 3x^2yzab^2 + (-x^3z + xz^3)b^3 + 3xyz^2a^2c - 3xyz^2b^2c - 3x^2yzac^2 + 3xy^2zbc^2 + (x^3y - xy^3)c^3.$$

Then we get C (black) and its BMSS dual curve D (red):



(**Problem**: Explain BMSS duality in terms of failure of SLP.)

Set up: Unexpected hypersurfaces

More recent work by various people, to be mentioned in more detail tomorrow by Juan Migliore, have extended the concept of unexpected curves to hypersurfaces.

Let $Z \subset \mathbb{P}^r$ be any variety and let $p \in \mathbb{P}^r$ be a general point.

One expects vanishing at p to order m to impose $\binom{m+r-1}{r}$ conditions.

Definition: We say I has an *unexpected* hypersurface of degree d if

$$\dim[I(Z) \cap (I(p)^m)]_d > \max\left(0, \dim[I(Z)]_d - \binom{m+r-1}{r}\right).$$

I.e., there are more hypersurfaces of degree d with a general point of multiplicity m and containing Z than expected.

CHMN Theorems

The following result relates failures of SLP and occurrence of unexpected plane curves.

Theorem (CHMN): A finite set of points $\mathcal{P}$ in $\mathbb{P}^2$ admits an unexpected curve of degree t if and only if $A = R/I(L_i^t : L_i \in \mathcal{L}_{\mathcal{P}})$ fails SLP in range 2 and degree $t-2$.

The following result characterizes occurrence of unexpected curves (and hence failures of SLP) but does not classify them. Here $m_{\mathcal{P}}$ is the least degree such that there is a curve of degree $m_{\mathcal{P}}+1$ vanishing on $\mathcal{P}$ with a general point of multiplicity $m_{\mathcal{P}}$.

Theorem (CHMN): A finite set of points $\mathcal{P}$ in $\mathbb{P}^2$ admits an unexpected curve if and only if $2m_{\mathcal{P}}+2 < |\mathcal{P}|$ but no subset of $m_{\mathcal{P}}+2$ or more of the points of $\mathcal{P}$ are collinear.

Supersolvable line arrangements

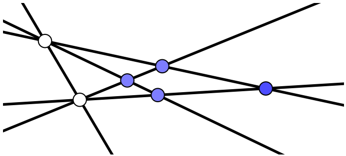
When $\mathcal{L}_{\mathcal{P}}$ is *supersolvable*, the previous theorem statement takes on a very nice form. First, some terminology.

Consider a line arrangement $\mathcal{L}$. Let $\mathcal{C}_{\mathcal{L}}$ be the set of *crossing points* for $\mathcal{L}$ (i.e., the points where 2 or more lines in $\mathcal{L}$ cross).

A point $p \in \mathcal{C}_{\mathcal{L}}$ is *modular* if for every $q \in \mathcal{C}_{\mathcal{L}}$, $q \neq p$, it is true that the line $\overline{pq}$ is in $\mathcal{L}$.

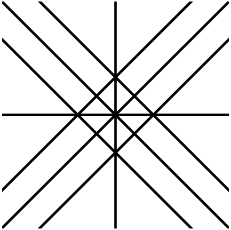
Definition: $\mathcal{L}$ is *supersolvable* (ss) if it has a modular point.

Example 1: This $\mathcal{L}$ has 5 lines and two modular points (white).



Example 2: B_3

The B_3 arrangement is supersolvable, with 3 modular points (two at infinity).



Theorem of Di Marca, Malara and Oneto (DMO)

Theorem (DMO: J. Alg. Comb., 2019) Let $\mathcal{L} = \{L_1, \dots, L_r\}$ be supersolvable, $m_{\mathcal{L}}$ the maximum multiplicity among the crossing points, and $d_{\mathcal{L}} = r$ the number of lines. The following are equivalent:

(a) $R/(L_1^d, \dots, L_r^d)$ fails SLP in range 2 and degree $d-2$ for some d;
(a′) $R/(L_1^d, \dots, L_r^d)$ fails SLP in range 2 and degree $d-2$ for $d = m_{\mathcal{L}}$;
(b) $\mathcal{P}_{\mathcal{L}}$ has an unexpected curve of degree d for some d;
(b′) $\mathcal{P}_{\mathcal{L}}$ has an unexpected curve of degree d for $d = m_{\mathcal{L}}$; and
(c) $2m_{\mathcal{L}} < d_{\mathcal{L}}$.

Question: Can we classify supersolvable $\mathcal{L}$ for which $R/(L_1^d, \dots, L_r^d)$ fails SLP in range 2 and degree $d-2$ for some d?

Question: I.e., which supersolvable $\mathcal{L}$ have $2m_{\mathcal{L}} < d_{\mathcal{L}}$?

Question: Can we classify supersolvable $\mathcal{L}$?

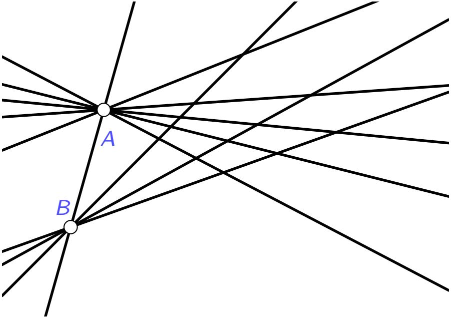
Classifying complex supersolvable $\mathcal{L}$

Based on 2019 results of Hanumanthu-BH (HH) and Dimca (D).

Let $\mathcal{L}$ be supersolvable (ss).

Definition: $\mathcal{L}$ is homogeneous (homog) if every modular point has the same multiplicity.

Theorem (HH): Let $\mathcal{L}$ be ss, non-homog. Then $\mathcal{L}$ looks like the following (and thus does not give a failure of SLP):



What you are seeing: there is a unique crossing point (here it's A) of maximum multiplicity $m_{\mathcal{L}}$, there is at most one other point (here it's B) of multiplicity more than 2, and all other crossing points have multiplicity 2.

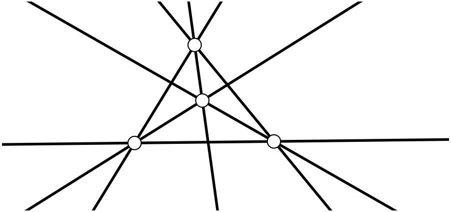
Proof: The key fact is a point of multiplicity $m_{\mathcal{L}}$ must be modular.

Homog ss $\mathcal{L}$

Theorem (HH): Let $\mathcal{L}$ be ss, homog. Then there are at most 4 modular points.

Proof: The key fact is no three modular points are collinear.

Theorem (HH): There is a unique homog $\mathcal{L}$ with 4 modular points; it has $m_{\mathcal{L}} = 3$:



Here we do not get a failure of SLP.

When are there 3 modular points?

Theorem (HH): If $\mathcal{L}$ has exactly 3 modular points, then $m_{\mathcal{L}} > 3$ and (up to choice of coordinates) the lines come from the linear factors of

$$xyz(x^{m-2} - y^{m-2})(x^{m-2} - z^{m-2})(y^{m-2} - z^{m-2}),$$

where $m = m_{\mathcal{L}}$. (Here the three modular points are the coordinate vertices.)

Notes:

(a) If $m = 3$ this gives the case of 4 modular points.

(b) Each with $m > 3$ gives a failure of SLP.

(c) The case $m = 4$ gives the B_3 arrangement.

A short-lived conjecture

Note: The 3 modular point ss line arrangement $\mathcal{L}$ given by

$$xyz(x^{m-2} - y^{m-2})(x^{m-2} - z^{m-2})(y^{m-2} - z^{m-2})$$

has $m_{\mathcal{L}} = m$ and $d_{\mathcal{L}} = 3(m_{\mathcal{L}} - 1)$ lines.

Conjecture (Dimca, July 2019)
A complex ss line arrangement $\mathcal{L}$ always has $d_{\mathcal{L}} \leq 3(m_{\mathcal{L}} - 1)$.

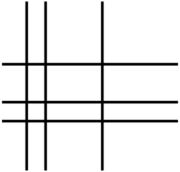
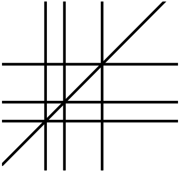
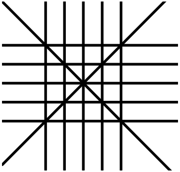
Theorem (Dimca and Abe, last week)
A complex ss line arrangement $\mathcal{L}$ always has $d_{\mathcal{L}} \leq 3(m_{\mathcal{L}} - 1)$.
This bound is sharp.

Two modular points, over the reals

Theorem (HH): A real homog $\mathcal{L}$ with exactly two modular points, has $d = 2t + 1 + \varepsilon$ with $m_{\mathcal{L}} = t + 1$ for $t \geq 1$ and $0 \leq \varepsilon \leq 2$, with the following excluded cases:
(a) t even, $\varepsilon = 2$ (these are not ss);
(b) $t = 2$, $\varepsilon = 1$ (this gives 4 modular points); and
(c) $t = 3$, $\varepsilon = 2$ (this gives 3 modular points).

Here are some $\mathcal{L}$ with just two modular points (which are at infinity; the line at infinity is not shown but is included in $\mathcal{L}$).

$\varepsilon = 0$ (7 lines, left), $\varepsilon = 1$ (8 lines, middle), $\varepsilon = 2$ (13 lines, right)

Two modular points, over the complexes

The ss $\mathcal{L}$ with exactly 2 modular points over the complexes are similar, except (up to lattice-isotopy) ε can be bigger: the range is

$$0 \leq \varepsilon \leq m_{\mathcal{L}} - 3.$$

Example: For $m > 1$, each of the following defines a ss $\mathcal{L}$ with exactly 2 modular points; here $0 \leq \varepsilon \leq m_{\mathcal{L}} - 3$, $m_{\mathcal{L}} = m + 1$ and $d_{\mathcal{L}} = 2m_{\mathcal{L}} - 1 + \varepsilon$:

$xyz(x^{m-1} - y^{m-1})(x^{m-1} - z^{m-1})$ (here $\varepsilon = 0$); and in general

$xyz(x^{m-1} - y^{m-1})(x^{m-1} - z^{m-1})(y^{m-1} - z^{m-1})H$, where H is the product of any $\varepsilon < m - 1$ factors of $y^{m-1} - z^{m-1}$ (here $\varepsilon = \deg H$).

Those with $\varepsilon \geq 2$ give rise to failures of SLP.

Theorem (D, 2019): Every complex ss $\mathcal{L}$ with exactly 2 modular points is lattice-isotopic to one of those above.

Open Problems and Conjectures

Given a line arrangement $\mathcal{L}$, let $t_k(\mathcal{L})$ denote the number of crossing points of multiplicity exactly k.

Conjecture (Anzis-Tohăneanu, 2016): A complex ss $\mathcal{L}$ has $2t_2(\mathcal{L}) \geq d_{\mathcal{L}}$. (Tohăneanu proved this over the reals (2014).)

Conjecture (Hanumanthu-BH, 2015): A complex ss $\mathcal{L}$ has $t_2(\mathcal{L}) > 0$.

By the results of Hanumanthu-BH and Dimca, these conjectures are known if $\mathcal{L}$ has at least 2 modular points.

Open Problem (Hanumanthu-BH, posed in 2019): Classify all complex $\mathcal{L}$, ss or not, with $2t_2(\mathcal{L}) < d_{\mathcal{L}}$.

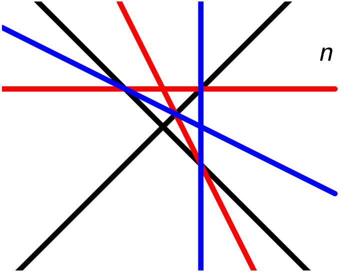
Open Problem: Classify all complex $\mathcal{L}$ with $t_2(\mathcal{L}) = 0$.

Only 4 complex $\mathcal{L}$ with $t_2(\mathcal{L}) = 0$ known: are there more?

(1) the arrangements $\mathcal{L}$ with $d \geq 3$ concurrent lines: $t_d(\mathcal{L}) = 1$, and $t_k(\mathcal{L}) = 0$ otherwise

(2) the arrangements $\mathcal{L}$ defined for $n \geq 3$ by the factors of $(x^n - y^n)(x^n - z^n)(y^n - z^n)$:

$d_{\mathcal{L}} = 3n$, $t_k(\mathcal{L}) = 0$ except for $t_3(\mathcal{L}) = 12$ if $n = 3$ and $t_3(\mathcal{L}) = n^2$, $t_n(\mathcal{L}) = 3$ for $n > 3$



$n = 2$

n concurrent black lines

so a point of multiplicity n

n concurrent red lines

so n^2 points where both

colors cross

n concurrent blue lines

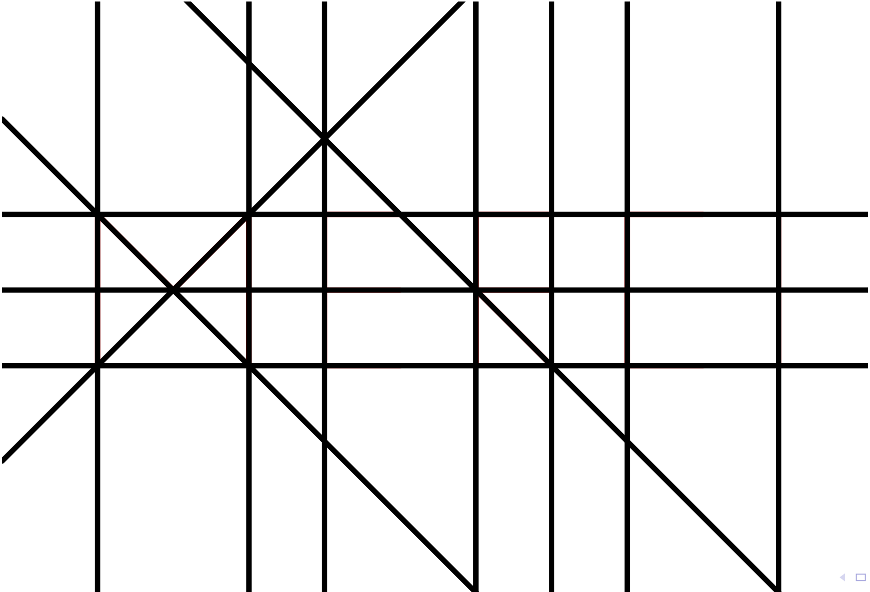
so $t_n = 1+1+1 = 3$ and $t_3 = n^2$

Two more are known

(3) an example $\mathcal{L}$ due to F. Klein (1879):
$d_{\mathcal{L}} = 21, t_3(\mathcal{L}) = 28, t_4(\mathcal{L}) = 21,$
and otherwise $t_k(\mathcal{L}) = 0$

(4) and an example due to A. Wiman (1896):
$d_{\mathcal{L}} = 45, t_3(\mathcal{L}) = 120, t_4(\mathcal{L}) = 45, t_5(\mathcal{L}) = 36,$
and otherwise $t_k(\mathcal{L}) = 0$

One more thing...



One more thing...