H-constants and Line Arrangements

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What are H-constants?

First introduced at MFO in 2010.

Goal: Study how singular a reduced plane algebraic curve can be.

Definition: Let $C \subset \mathbb{P}^2$ be a reduced singular curve:

$$d = \deg(C)$$

$$\{p_1, \ldots, p_r\} = \mathrm{Sing}(C)$$

$$m_i = \mathrm{mult}_{p_i}(C).$$

Then

$$H(C) = \frac{d^2 - (m_1^2 + \cdots + m_r^2)}{r}.$$

Relation to Bounded Negativity Conjecture (BNC)

BNC (rational case): Let X be a smooth projective rational surface over a field $k = \overline{k}$.

(1) $\exists\ b_X$ such that $C^2 \geq b_X\ \forall$ reduced, irreducible curves $C \subset X$.

(2) $\exists\ B_X$ such that $C^2 \geq B_X\ \forall$ reduced curves $C \subset X$.

Theorem: $\inf\limits_{C \text{ red}} H(C) > -\infty \Rightarrow \text{BNC } 1 \iff \text{BNC } 2$.

First implication: obvious.

Second implication, $\Leftarrow$: obvious

Second implication, $\Rightarrow$: **Reference**: Negative curves on algebraic surfaces, Duke Math J. 162 (2013) by Bauer, Harbourne, Knutsen, Küronya, Müller-Stach, Roulleau & Szemberg.

What is known about the values of H-constants?

- No example is known of a reduced & irreducible C with $H(C) \leq -2$.

However, if C_d is a general plane rational curve of degree d, then

$$H(C_d) = -2 + \frac{6d-4}{(d-1)(d-2)}$$

so $\inf_{\substack{C \text{ reduced,} \\ \text{irred., sing.}}} H(C) \leq -2$.

- No example is known of a reduced C over $\mathbb{C}$ with $H(C) \leq -4$.

However, examples show $\inf_{\substack{C \text{ reduced,} \\ \text{singular}}} H(C) \leq -4$.

Reference: Bounded negativity, Miyaoka-Sakai inequality and elliptic curve configurations, IMRN (2017) by Roulleau.

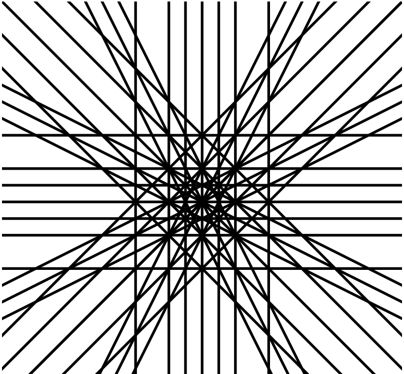
What if C is a union of lines?

Open Question: What is the most negative value of $H(C)$ when C is a rational arrangement of lines?

The most negative such example C known is a simplicial arrangement of 37 lines ($\mathcal{A}(37,3)$ in Grünbaum's list; recall Michael Cuntz's discussion):

$$H(C) = -\frac{503}{181} = -2.78.$$

$\mathcal{A}(37,3)$ is shown at right; the 37th line is at infinity.



What if C is a union of real or complex lines?

In the real case, $H(C) > -3$ but $\inf H(C) = -3$, using simplicial, supersolvable examples.

Open Question: How negative can $H(C)$ be for a complex line arrangement?

Theorem: For complex line arrangements, $H(C) > -4$.

The most negative known example comes from a curve with 45 lines constructed by A. Wiman (1896). It has

$$H(C) = -\frac{225}{67} = -3.56.$$

Reference: Bounded Negativity and Arrangements of Lines, IMRN (2015) by Bauer, Di Rocco, Harbourne, Huizenga, Lundman, Pokora & Szemberg

Open problems for supersolvable (ss) line arrangements.

Open Problem: How negative can $H(C)$ be when C is complex and supersolvable? (Is it true that $H(C) > -3$?)

Recall: A singular point p of a line arrangement C is *modular* if every other singular point of C is on a component of C through p.

We say C is *supersolvable* if it has one or more modular points.

Possible approach: Classify the complex supersolvable C.

Theorem: A "nontrivial" complex supersolvable C has at most 4 modular points. Moreover, those with more than one modular point have been classified, and these have $H(C) > -3$.

Open Problem: Classify complex supersolvable arrangements with only one modular point.

References: (1) Real and complex ss line arrangements in the projective plane, to appear, J. Alg. by Hanumanthu & Harbourne
(2) On complex ss line arrangements, J. Alg. (2020) by Abe & Dimca

Simple crossings

Note: When C is a line arrangement, $H(C) = \dfrac{d^2 - \sum_{i=1}^r m_i^2}{r}$ simplifies to $H(C) = \dfrac{d - \sum m_i}{r} = \dfrac{d}{r} - \overline{m}$.

Thus having any $m_i = 2$ seems bad if you want very negative $H(C)$.

Conjecture (Anzis and Tohăneanu; now a Theorem of Abe): Let t_2 be the number of points of multiplicity 2. For a complex supersolvable C of d lines, we have $t_2 \geq d/2$.

References:

(1) On the geometry of real or complex supersolvable line arrangements, J. Combin. Theory (2016) by Anzis & Tohăneanu

(2) Double points of free projective line arrangements, IMRN 2020 by Abe.

Final open problem

The Wiman arrangement of 45 lines has 120 points of multiplicity 3, 45 of multiplicity 4 and 36 of multiplicity 5, but none of multiplicity 2 (i.e., $t_2 = 0$).

Maybe others with $t_2 = 0$ would give very negative H-constants.

Open Problem: Classify complex line arrangements with $t_2 = 0$.

Only 4 kinds are currently known:

(1) Trivial cases ($n \geq 3$ concurrent lines)

(2) Fermat: $(x^n - y^n)(x^n - z^n)(y^n - z^n) = 0$ with $n \geq 3$

(3) Klein's (1879): 21 lines with 21 points of multiplicity 4 and 28 of multiplicity 3

(4) Wiman's (1896).

Thank you for your attention!