Geproci sets: a confluence of geometry, algebra, combinatorics and representation theory

Brian Harbourne

Cather Professor of Mathematics
Fellow of the AMS
Department of Mathematics
University of Nebraska-Lincoln (UNL)

Algebra and Algebraic Geometry Seminar, McMaster Univ.:
January 26, 2023

Slides will be available at my website (green text is clickable):
https://www.math.unl.edu/~bharbourne1/

Main reference: arXiv:2209.04820

This talk is on joint work of the POLITUS Team (PT):

Luca
Chiantini

Łucja
Farnik

Giuseppe
Favacchio

Brian
Harbourne

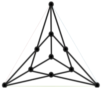
Juan
Migliore

Tomasz
Szemberg

Justyna
Szpond

POLITUS: POLand, ITaly and the US

An international collaboration of 7 researchers whose logo is a stylized D_4 configuration:



Connections

Our work has connections to:

Algebraic Geometry

Commutative Algebra

Combinatorics

Representation Theory

Quantum Physics

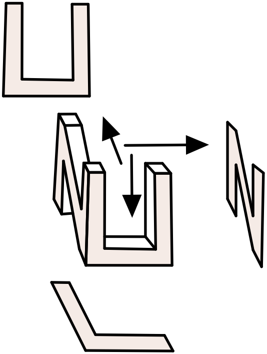
It can also be considered in a larger not-only-mathematical context!

Abstract

Studying inverse scattering problems has led to remarkable advances in scientific knowledge. Here we propose carrying this idea over to classification problems in algebraic geometry. As a specific example we consider the following problem: which finite subsets of 3 dimensional projective space project from a general point to a complete intersection in a plane?

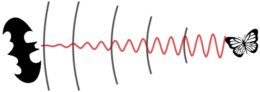
Inverse scattering Problems (ISP): try to discern structure from projected or reflected data.

Idea: classify structures algebro-geometrically based on properties of projected images.

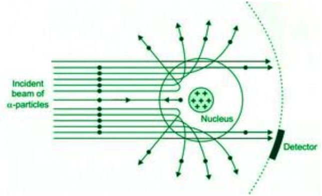


Some examples of ISP

Echolocation (biology):

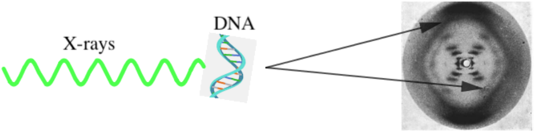


Rutherford scattering (physics; led to Bohr model of atom):



X-ray crystallography (chem/bio; led to DNA double helix model):

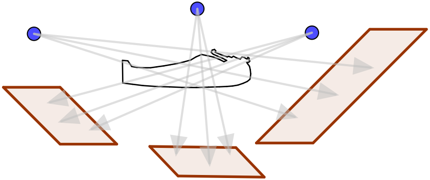
X-ray diffraction pattern



Rosalind Franklin
"Photo 51"

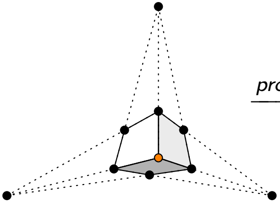
More examples

Tomography (medicine):

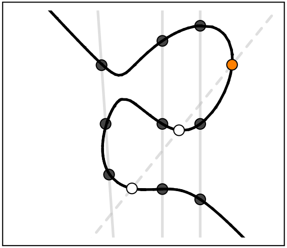


GePro-$\mathcal{P}$ (math): Pick a property $\mathcal{P}$ and classify finite point sets in $\mathbb{P}^n$ whose Ge neral Pro jections to a hyperplane satisfy $\boxed{\mathcal{P}}$.

Here are 12 points (10 visible) in space whose projections from general points to a plane are complete intersections (so $\mathcal{P}$ is "being a CI"). These 12 points (known as D_4) are "geproci."

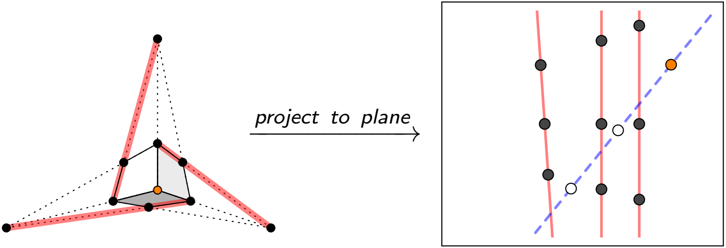


project to plane

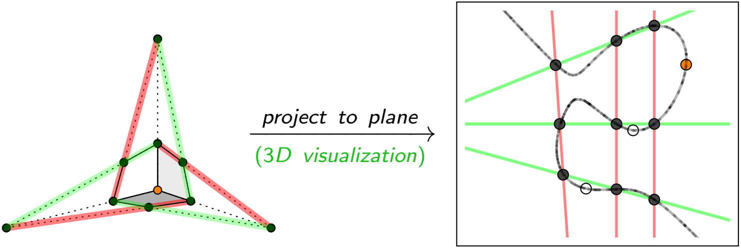


Why D_4 is geproci

The quartic comes from lines through collinear points:



The cubic is one in a pencil of cubics:



Gepro-$\mathcal{P}$

General Problem: Given a property $\mathcal{P}$ of finite point sets $\overline{Z} \subset \mathbb{P}^{n-1}$, classify all finite $Z \subset \mathbb{P}^n$ such that $\overline{Z} \subset H \cong \mathbb{P}^{n-1}$ has property $\mathcal{P}$ (where $\overline{Z}$ is the image of Z under projection $\mathbb{P}^n \dashrightarrow H$ from a general point P to a hyperplane $H \subset \mathbb{P}^n$).

Example 1: Say $\mathcal{P}$ means "$\overline{Z}$ is Gorenstein". Then a set Z of $n+1$ general points in $\mathbb{P}^n$ is gepro-$\mathcal{P}$ since the image $\overline{Z}$ is a set of $n+1$ general points in H, which is Gorenstein.

Open Problem 1: Classify gepro-Gorenstein sets Z.

Every geproci set is also gepro-Gorenstein but not conversely.

Open Problem 2: Classify geproci sets in $\mathbb{P}^n$.

History of geproci

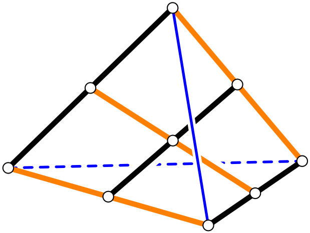
We know no interesting examples of geproci sets in $\mathbb{P}^n$ for $n > 3$.

We say $Z \subset \mathbb{P}^3$ is (a, b)-geproci if $\overline{Z}$ is the intersection of a curve A of degree a with a curve B of degree b, with $a \leq b$.

A geproci set Z in a plane $H \subset \mathbb{P}^3$ is called *degenerate*; it is just the complete intersection of two curves in H.

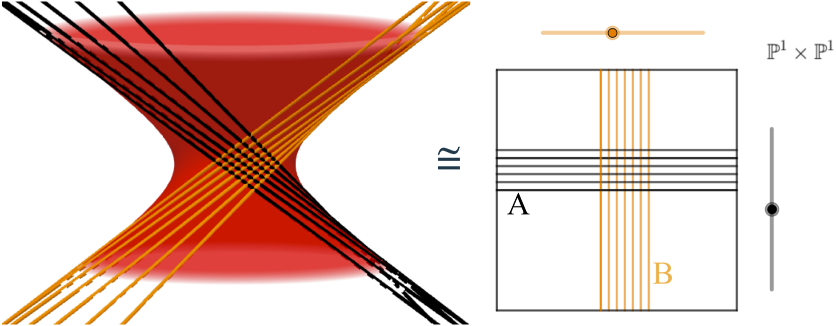
Question 1 (F. Polizzi 2011): Is every geproci set in $\mathbb{P}^3$ degenerate?

Answer (D. Panov, 2011): No!
(a, b)-grids are nondegenerate and geproci. I.e., $2 \leq a \leq b$ with A being a skew black lines and B being b skew orange lines, where each black line intersects each orange line in exactly 1 point. (Here $a = b = 3$.)



We understand grids.

Fact: For an (a, b)-grid with $3 \leq a \leq b$, the grid lines come from the rulings on a smooth quadric.



Fact: A $(2, b)$-grid consists of b points on each of two skew lines (but the grid lines need not all lie on a smooth quadric).

New Question and a partial answer

For simplicity, call a geproci set in $\mathbb{P}^3$ *trivial* if it is either a grid or contained in a plane.

Question 1′ (F. Polizzi 2011): Is every geproci $Z \subset \mathbb{P}^3$ trivial? If not, can such Z be classified up to projective equivalence, at least when $|Z|$ is small?

Answer (2018, Lefschetz Working Group at Levico Terme): Certain finite sets Z given by root systems (such as D_4 and F_4) which have unexpected cones (see Harbourne-Migliore-Nagel-Teitler: arXiv:1805.10626, Michigan Math. J. 2020) turn out to be nontrivial geproci sets.

Theorem (Levico Terme Working Group, 2018): A finite set $Z \subset \mathbb{P}^3$ is (a, b)-geproci if $|Z| = ab$ and it has unexpected cones of degrees $a \leq b$ with no common components.

The 2018 Levico Terme Working Group (LTWG)

Alessandra Bernardi

Luca Chiantini

Graham Denham

Giuseppe Favacchio

Brian Harbourne

Juan Migliore

Tomasz Szemberg

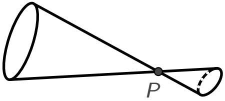
Justyna Szpond

Unexpected cones

Let $p_1, \ldots, p_s \in \mathbb{P}^n$ be distinct and let $P \in \mathbb{P}^n$ be general.

Let $Z = \{p_1, \ldots, p_s\}$ and $I(Z) \subset k[\mathbb{P}^n] = k[x_0, \ldots, x_n]$ its ideal.

The cones of degree t with vertex P: $[I(P)^t]_t$



The cones of degree t with vertex P containing Z: $[I(Z)]_t \cap [I(P)^t]_t$

They're "unexpected" if there are more than expected:

$$\dim\left([I(Z)]_t \cap [I(P)^t]_t\right) > \max\left\{0, \dim[I(Z)]_t - \binom{n+t-1}{n}\right\}.$$

We write "Z satisfies $C(t)$" if Z has unexpected cones of degree t.

A result and an Open Problem

Theorem (Chiantini-Migliore, arXiv:1904.02047, TAMS 2021): Every (a, b)-grid with $3 \leq a \leq b$ satisfies both $C(a)$ and $C(b)$.

Open Problem 3: Does every nontrivial (a, b)-geproci $Z \subset \mathbb{P}^3$ satisfy both $C(a)$ and $C(b)$?

D_4 played a special role; F_4 was important too!

Theorem (Chiantini-Migliore, arXiv:1904.02047, TAMS 2021): The least $|Z|$ for a nontrivial geproci set is $|Z| = 12$. An example is given by the D_4 configuration of 12 points; it is $(3, 4)$-geproci.

Theorem 1 (PT, 2022): The D_4 configuration is, up to projective equivalence, the only nontrivial (a, b)-geproci set in $\mathbb{P}^3$ with $a \leq 3$.

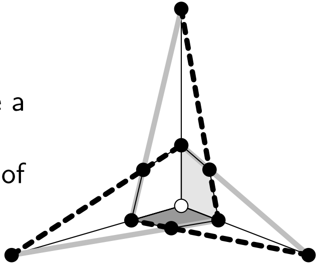
D_4 and F_4 motivated the following theorem:

Theorem 2 (PT, 2022): For each $4 \leq a \leq b$, there is a nontrivial (a, b)-geproci $Z \subset \mathbb{P}^3$.

The proof of Theorem 2 starts with specific (a, a)-grids and adds one (or two) specific set(s) of a collinear points, as exemplified by D_4 (or F_4), followed by deletions of certain collinear subsets.

How D_4 and F_4 motivated Theorem 2

D_4 is at right: the gray and dashed lines give a $(3,3)$-grid. The main diagonal of the cube through the white point is the additional set of 3 collinear points.



F_4 is the 24 intersection points of the $\binom{8}{2} = 28$ lines through pairs of vertices of a cube.

Fact: It has unexpected cones of degrees 4 and 6, and $|F_4| = 24$.

Conclusion (LTWG): F_4 is a $(4,6)$-geproci set.

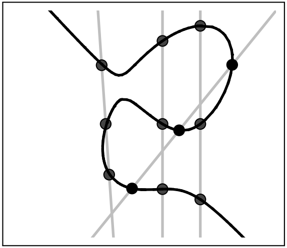
For F_4, $\overline{Z}$ is the intersection of 6 lines with an irreducible quartic.

Here is a 3D view of F_4, its $(4,4)$-subgrid and its unexpected cones.

D_4 and F_4 are half grids

Definition: A nontrivial (a, b)-geproci set Z is a *half grid* if $\overline{Z}$ is the intersection of two curves, exactly one of which can always be taken to be a union of lines.

Example: $Z = D_4$ is not contained in a smooth quadric so it is not a grid. Here $\overline{Z}$ is contained in 4 lines, so it is a half grid.



Example: $Z = F_4$ is not contained in a smooth quadric so it is not a grid. Here $\overline{Z}$ is contained in 6 lines, so it is a half grid.

Open Problems

(a) We know only a few examples of nontrivial geproci non-half grids:
- The 60 point set for the H_4 root system (Wiśniewska-Zięba).
- A 40 point $(5, 8)$-geproci set applied by Penrose to quantum mechanics (QM).
- A 120 point $(10, 12)$-geproci set also related to QM.

Open Problem 4: Are there only finitely many nontrivial geproci non-half grids?

(b) Every nontrivial geproci set in $\mathbb{P}^3$ that we know of has multiple subsets of at least 3 collinear points.

Open Problem 5: Can a nontrivial geproci set be linearly general?

(c) The 40 point Penrose set is Gorenstein.

Open Problem 6: Are there other finite Gorenstein geproci sets?

More Open Problems

(a) There are, up to projective equivalence, uncountably many grids.

Open Problem 7: Up to projective equivalence, is there any (a, b) with infinitely many nontrivial (a, b)-geproci sets?

(b) We know no example of a geproci set in $\mathbb{P}^n$ for $n > 3$.

Open Problem 8: Do geproci sets exist in $\mathbb{P}^n$, $n > 3$?

(c) We can define a geproci variety as any variety whose general projection is a complete intersection. A cone with a general vertex over a finite geproci set is a geproci curve, the cone over that is a geproci surface, etc. These geproci varieties all have codimension 3.

Open Problem 9: Are there other kinds of geproci varieties? Are there any with codimension greater than 3?

Terao type problems

Terao's Conjecture concerns whether a certain property of hyperplane arrangements is a combinatorial property. A geproci set also has combinatorics (e.g., its collinear subsets).

Open Problem 10: If two geproci sets have the same combinatorics, are they projectively equivalent?

Open Problem 11: If a set has the same combinatorics as a geproci set, is it geproci?

Other work: Exploring geproci in positive characteristics.

Theorem [Jake Kettinger]: Let k be a finite field, $q = |k|$, $\overline{k}$ its algebraic closure and let $Z = \mathbb{P}^3_k$ be all k-points of $\mathbb{P}^3_{\overline{k}}$. Then Z is a nontrivial $(q+1, q^2+1)$-geproci set.

Proof: Uses the fact that $\mathbb{P}^3_k$ has a full *spread*; i.e., a decomposition into q^2+1 disjoint sets of $q+1$ collinear points.

Definition: A partial spread for k consists of $b < q^2+1$ disjoint sets each with $q+1$ collinear k-rational points. A *maximal partial spread* is a partial spread maximal with respect to inclusion.

Theorem [Jake Kettinger]: Let k be a finite field, $q = |k|$, $\overline{k}$ its algebraic closure and let Z be the complement in $\mathbb{P}^3_k$ of a maximal partial spread. Then Z is a nontrivial non-halfgrid $\{q+1, b\}$-geproci set (where $\{q+1, b\} = (q+1, b)$ if $q+1 < b$ and $\{q+1, b\} = (b, q+1)$ if $b \leq q+1$).

Thanks for your attention!

FINIS