The concept of geproci subsets of $\mathbb{P}^3$

Brian Harbourne

Cather Professor of Mathematics
2022 Fellow of the AMS
Department of Mathematics
University of Nebraska-Lincoln

Algebraic Geometry and Commutative Algebra Seminar

University of Notre Dame

November 9, 2021

Slides will be available at my website:
`https://www.math.unl.edu/~bharbourne1/`

Timeline (in years before present)

- t=-139 Emmy Noether, born 1882
- t=-95 Grete Hermann, Noether's 1st student receives PhD (her 1926 thesis laid foundation for computer algebra)
- t=-89 John von Neumann, proved impossibility of hidden variables in quantum mechanics in 1932
- t=-55 John Stewart Bell showed von Neumann's proof did not show what was claimed (Bell's 1966 Theorem)
- t=-47 In 1974 Max Jammer pointed out Hermann had in 1935 already raised the issue Bell addressed (but was largely ignored)
- t=-5 CHMN: Cook, H___, Migliore and Nagel introduced the notion of "unexpected curves" (preprint: arXiv:1602.02300; appeared as *Line arrangements and configurations of points with an unexpected geometric property*, Compositio Math. 154:10 (2018) 2150–2194)
- t=-3.67 Ground zero: 3-9-2018 I spoke here on unexpected curves. An audience comment has led to an explosion of research.

Recall: unexpected curves.

$Z \subset \mathbb{P}^2$: a finite set of points.

$I(Z) \subset \mathbb{C}[x, y, z] = \mathbb{C}[\mathbb{P}^2]$: the ideal of forms vanishing on Z.

$[I(Z)]_t$: vector space span of forms in $I(Z)$ of degree t.

For any point $P \notin Z$ and multiplicity $m = t - 1$ we have

$$\dim[I(Z) \cap I(P)^m]_t \geq \max(0, \dim[I(Z)]_t - \binom{m+1}{2}).$$

But for P general we typically "expect"

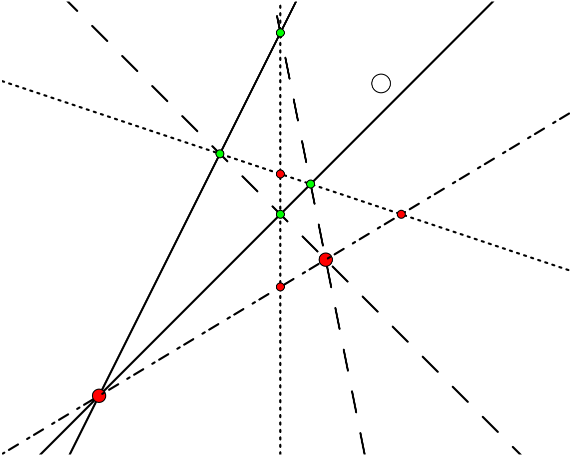
$$\dim[I(Z) \cap I(P)^m]_t = \max(0, \dim[I(Z)]_t - \binom{m+1}{2}).$$

So we say the curves defined by $[I(Z) \cap I(P)^m]_t$ are *unexpected* if $\dim[I(Z) \cap I(P)^m]_t > \max(0, \dim[I(Z)]_t - \binom{m+1}{2})$.

We also say Z has unexpected curves of degree t with a general point P of multiplicity m.

The example I gave.

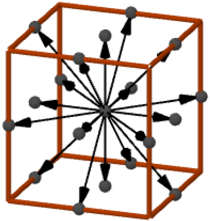
The simplest possible example is a set Z of 9 points with a unique unexpected quartic with $m = 3$. I.e., there is a unique unexpected quartic through the 9 points of Z, singular with multiplicity 3 at the general point:



Other examples.

CHMN also gave other examples coming from line arrangements, found on a more or less random basis.

But at my 2018 talk here, Matthew Dyer pointed out that the previous 9 point example Z was the projectivization of the B_3 root system:



Here is a 3D view.

Consequences: Dyer's comment led to the paper HMNT.

B. Harbourne, J. Migliore, U. Nagel, Z. Teitler.
Unexpected hypersurfaces and where to find them,
Mich. Math. J., 2021 (arXiv:1805.10626).

HMNT extended unexpected curves to unexpected hypersurfaces:

For a finite point set $Z \subset \mathbb{P}^n$, a degree t, a multiplicity $m \leq t$ and a general point $P \in \mathbb{P}^n$, we say the hypersurfaces defined by $[I(Z) \cap I(P)^m]_t$ are *unexpected* if

$$\dim[I(Z) \cap I(P)^m]_t > \max(0, \dim[I(Z)]_t - \binom{m+n-1}{n}).$$

If $m = t$, the unexpected hypersurface is a cone.

HMNT gives examples of point sets Z_R with unexpected hypersurface cones in various $\mathbb{P}^n$ coming from root systems R.

$t = -3.5$: New HMNT examples

$R = B_{n+1}$: $Z_R \subset \mathbb{P}^n$ has unexpected cones of degree 4 for $n = 3, 4$.

$R = B_{n+1}$: $Z_R \subset \mathbb{P}^n$ has unexpected cones of degrees 3, 4 for $n = 5, 6$.

$R = D_4$: $Z_R \subset \mathbb{P}^3$ has unexpected cones of degrees 3 and 4.

$R = E_7$: $Z_R \subset \mathbb{P}^6$ has unexpected cones of degree 4.

$R = E_8$: $Z_R \subset \mathbb{P}^7$ has unexpected cones of degrees 4 and 5.

$R = F_4$: $Z_R \subset \mathbb{P}^3$ has unexpected cones of degrees 4, 5, 6 and 7.

$R = H_3$: $Z_R \subset \mathbb{P}^2$ has unexpected curves of degrees 6, 7 and 8.

$R = H_4$: $Z_R \subset \mathbb{P}^3$ has unexpected cone of degree 6 (later P. Fraś, M. Zięba, arXiv:2107.08107, showed it had another one of degree 10).

$t = -3$: Workshop at Levico Terme in 2018

A working group at Levico Terme noticed something interesting for some of these Z_R in $\mathbb{P}^3$:

$R = D_4$: $|Z_R| = 12$ has unexpected cones of degrees 3 and 4.

$R = F_4$: $|Z_R| = 24$ has unexpected cones of degrees 4 and 6.

Fact (Workshop working group at Levico Terme, 2018): If $|Z| = ab$ has unexpected cones of degrees a and b with no components in common, then the projection of $Z \subset \mathbb{P}^3$ from a general point to a plane is the complete intersection of curves of degrees a and b (i.e., Z is (a, b)-geproci: its GEneral PROjection to a plane is a Complete Intersection).

This is written up in ACM, i.e., in

Appendix to Chiantini-Migliore: Trans. AMS, 2021 (arXiv:1904.02047), "Sets of points which project to complete intersections."

(Note: The Levico workshop led to the Chiantini-Migliore paper.)

The 2018 Levico Terme working group

Alessandra Bernardi

Luca Chiantini

Graham Denham

Giuseppe Favacchio

Brian Harbourne

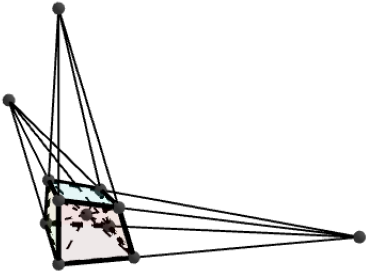
Juan Migliore

Tomasz Szemberg

Justyna Szpond

Let's look at $R = D_4$.

Z_{D_4} has 12 points and unexpected cones of degree 3 and 4. The 12 points come from a cube in 3 point perspective.



The quartic cone is easy to see. It is the cone with vertex P on 4 skew lines containing Z_{D_4}.

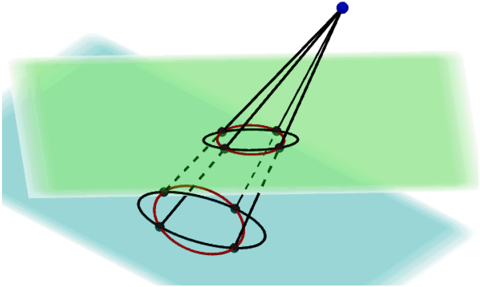
The cubic cone comes from a pencil defined by two cubic cones. Here are the two cubic cones. And here is the pencil of cubic cones).

$t = -10$: A question (6-8-2011).

In fact, unexpected cones unexpectedly answered a 2011 question!

Mathoverflow Quest. 67265 by Francesco Polizzi: When is a general projection of points in $\mathbb{P}^3$ a complete intersection? Are there nontrivial examples?

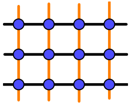
Trivial example: A complete intersection (CI) of two curves in the same plane projects isomorphically to its image, so is trivially geproci.



$t = -9.99988$: Answer by Dmitri Panov (6-8-2011): Grids!

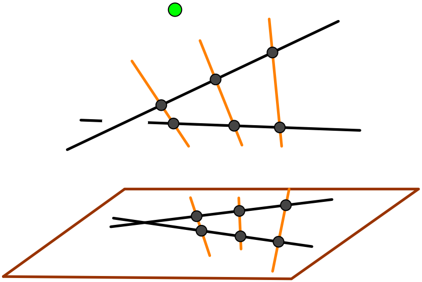
An (a, b)-*grid* is (a, b)-geproci. What is an (a, b)-grid?

It is given by a skew black lines and b skew orange lines, such that each black line meets each orange line in one point. The ab points form the grid. The lines are called *grid lines*.

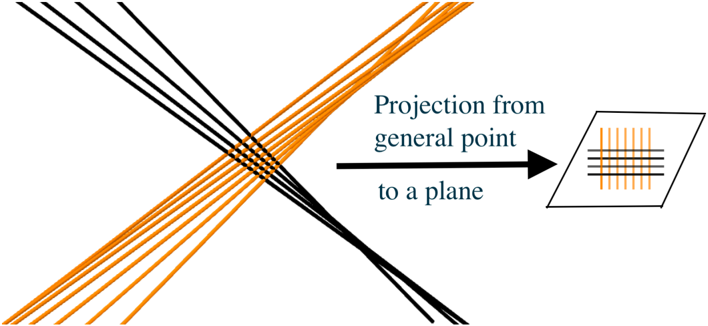


A (3, 4)-grid

Construction is easy when $a = 2$. Here's one with $(a, b) = (2, 3)$.

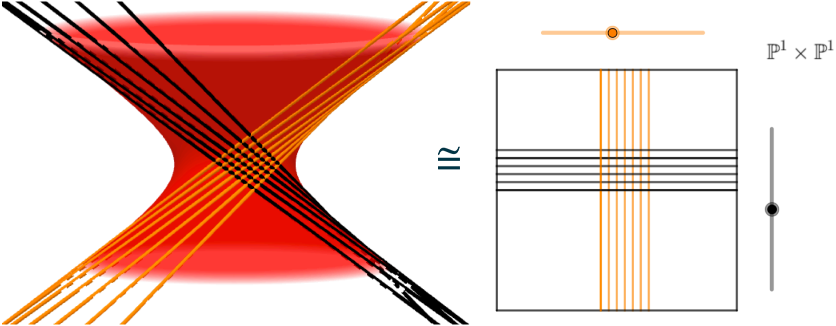


What about $a > 2$?



The graphic shows a $(4, 7)$-grid. It projects to a complete intersection of 4 lines with 7 lines, so it is $(4, 7)$-geproci. But where do such lines come from?

What about $a > 2$?



They come from the two rulings on a smooth quadric! And every grid with $2 < a \leq b$ works this way.

$t = -9.997$: Question edit (6-9-2011)

Based on Panov's construction, Polizzi edited his question:

(1) Are there nontrivial nongrid geproci sets?

(2) Can we classify them (at least for small numbers of points)?

Answers: (1) Yes (2018; e.g., Z_{D_4}) and (2) yes (on-going)!

(1) The first non-trivial non-grid geproci sets were found in 2018, based on both HMNT and the Levico Terme workshop:

Examples: Z_{D_4} is $(3, 4)$-geproci and Z_{F_4} is $(4, 6)$-geproci.

(Note: Z_{F_4} is the set of 24 intersection points of the $\binom{8}{2} = 28$ lines through pairs of vertices of a cube.)

Examples (ACM): Z_{F_4} contains Z_{D_4} and also two types of $(4, 4)$-geproci (a grid and a half-grid) and a half-grid $(4, 5)$-geproci. (Being a half-grid means exactly one of the unexpected cones can be taken to be a cone over a union of lines.)

$t = -1$: More examples and a start on classification

(1) Examples announced at an MFO workshop in October, 2020:

Example (P. Fraś, M. Zięba, arXiv:2107.08107): Z_{H_4} is a nontrivial, nongrid, non-half-grid $(6, 10)$-geproci.

Example (P. Pokora, T. Szemberg, J. Szpond, arXiv:2010.08863): A 60 point set due to Klein is a nontrivial $(6, 10)$-geproci half-grid.

(2) The Chiantini-Migliore paper also gave results on classification:

Theorem (Chiantini-Migliore TAMS 2021) All nontrivial nongrid geproci sets have at least 12 points (because nontrivial (a, b)-geproci sets with $2 = a \leq b$ or $a = b = 3$ are grids).

And new insights on grids:

Theorem (Chiantini-Migliore TAMS 2021): Any (a, b)-grid with $ab > 4$ has unexpected cones of degrees a and b.

$t < -1$: The Geproci Squad, results and questions

Levico and a 10-2020 MFO workshop led to forming the Geproci Squad to work on geproci questions. Here is some work in progress.

(1) **Theorem** (Geproci Squad): Given $4 \leq a \leq b$, there is a nontrivial half-grid (a, b)-geproci set $Z \subset \mathbb{P}^3$.

(2) **Theorem** (Geproci Squad) Z_{D_4} is the unique 12 point nontrivial nongrid geproci.

Some Questions:

(Q1) Is Z_{D_4} the only nontrivial nongrid $(3, b)$-geproci Z?

(Q2) Which a, b have a unique nontrivial nongrid (a, b)-geproci Z? Is $a = 3, b = 4$ (i.e., Z_{D_4}) the only one?

(Q3) Do all nontrivial (a, b)-geproci Z (except the $(2, 2)$-grid) come from unexpected cones of degrees a and b?

(Q4) Is Z_{H_4} the only nontrivial nongrid non-half-grid geproci set?

$t = -.016$: Late breaking news! (11-3-2021)

Some time ago Squad member Giuseppe Favacchio ran across the fact that Z_{D_4} had been used in giving proofs of Bell's Theorem.

November 3, 2021: So Giuseppe searched further and found Z_{F_4} and Z_{H_4} also had been used in giving proofs of Bell's Theorem. And moreover, yet another set of points, based on the Penrose Dodecahedron, was used to prove Bell's Theorem. It's a 40 point set which also turned out to be geproci: it is a nontrivial, nongrid, non-half grid $(5, 8)$-geproci set.

Thus the answer to (Q4) is: Z_{H_4} is not the only nontrivial nongrid non-half-grid geproci set!

Revised question (Q4): For which a and b is there a nontrivial nongrid non-half-grid (a, b)-geproci?

New larger question (Q5): What exactly is the connection to Quantum Mechanics?

The Geproci Squad

Luca
Chiantini

Giuseppe
Favacchio

Brian
Harbourne

Juan
Migliore

Tomasz
Szemberg

Justyna
Szpond

Łucja
Farnik