From SHGH to Geproci (and beyond)

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Organizing Committee: Elena Guardo, Alessandro Oneto, Elisa Postinghel, Pierpaola Santarsiero.

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Slides available eventually at my website (green text is clickable):
https://unlblh.github.io/BrianHarbourne/

Interpolation problems

R_d: all forms on $\mathbb{P}^n$ of degree d, so $R = \mathbb{C}[\mathbb{P}^n] = \oplus_d R_d$, $\dim R_d = \binom{d+n}{n}$.

$\overline{m}\overline{P} = m_1 P_1 + \cdots + m_s P_s \subset \mathbb{P}^n$: notation for scheme defined by all forms vanishing to order $\geq m_i$ at **general points** P_i, where $\overline{m} = (m_1, \ldots, m_s)$.

$[I(\overline{m}\overline{P})]_d = I(\overline{m}\overline{P}) \cap R_d$, so $I(\overline{m}\overline{P}) \subset R$ is the ideal of $\overline{m}\overline{P}$.

Problem: Given $\overline{m}$ and d, $\boxed{\text{find } \dim\ [I(\overline{m}\overline{P})]_d.}$

Theorem 1: $\dim[I(\overline{m}\overline{P})]_d \geq \max\left\{0, \dim R_d - \sum_{m_i>0} \binom{m_i+n-1}{2}\right\}$

Problem: When do we have "$>$"? I.e., $\dim[I(\overline{m}\overline{P})]_d > 0$ yet the points $\sum m_i P_i$ fail to impose independent conditions on R_d?

SHGH Conjecture: When $n = 2$, SHGH gives explicit description for when "$>$" occurs and what the value of $\dim[I(\overline{m}\overline{P})]_d$ is.

SHGH

B. Segre
1961

BH
1986

A. Gimigliano
1987

A. Hirschowitz
1989

- **Theorem**: All four versions are equivalent. (Ciliberto/Miranda, 2001)

Taking into account the known ways that "$>$" occurs, SHGH reduces to:

SHGH Conjecture: If $n = 2$, $d \geq m_1 + m_2 + m_3$ and $m_1 \geq \cdots \geq m_s$, then $\dim[I(\overline{m}\overline{P})]_d = \max\left\{0, \dim R_d - \sum_{m_i>0} \binom{m_i+1}{2}\right\}$.

Idea: Could it help to study versions of a more general problem?

New concept: Unexpectedness

Introduced by Cook, H., Migliore, Nagel, Compositio '18:

Say $\mathbb{P}^n$ has *unexpected hypersurfaces* for vector subspace $V \subseteq R_d$ and multiplicities $\overline{m} = (m_1, \ldots, m_s)$ if

$$\dim[V \cap I(\overline{m}\overline{P})]_d > \max\{0, \dim V - \sum_i \binom{m_i + n - 1}{n}\}.$$

Original SHGH problem: $n = 2$, $V = R_d$. The SHGH Conjecture, if true, means the only unexpectedness in this situation is what's already known.

But $V = R_d$ is too hard. Let's try something $V \subsetneq R_d$ easier!

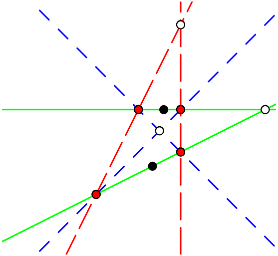
Let $V = [I(Z)]_d$ for finite set of reduced special points $Z \subset \mathbb{P}^n$,

$\overline{m}\overline{P} = mP \subset \mathbb{P}^n$ a single fat general point (so $s = 1$, $m = m_1$, $P = P_1$).

Problem: Understand n, Z, m, d where unexpectedness occurs.

CHMN

First example



Z is a set of 9 points in $\mathbb{P}^2$:

4 general points (red) giving a pencil of conics;

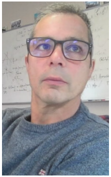
3 points (white), singular points of the singular conics through the red points;

2 points (black) where a singular conic meets the line through the singular points of the other 2 singular conics.

We found this set of points in Di Gennaro, Ilardi, Vallès, 2014:

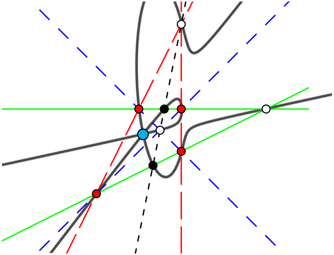
Giovanna Ilardi

Roberta Di Gennaro



Jean Vallès

Computations of unexpectedness:



(Click here for dynamic GeoGebra construction)

$$\dim[I(Z \cup 3P)]_4 > \max\{0, \dim[I(Z)]_4 - \binom{3+2-1}{2}\}$$

$$1 > \max\{0, 6 - 6\}$$

So Z has an unexpected quartic! Here it is in black (the general triple point P is in light blue).

This Z comes from a root system!

Root systems and unexpectedness

See H., Migliore, Nagel, Teitler, Mich. J. '21 for B_3, D_4, F_4:

Projectivizing the B_3 root system gives the 9 points; $Z = Z(B_3)$!

Projectivizing D_4 gives 12 points $Z(D_4) \subset \mathbb{P}^3$ with two unexpected cones of degrees 3 and 4 (note: $3 \cdot 4 = Z(D_4)$).

Projectivizing F_4 gives 24 points $Z(F_4) \subset \mathbb{P}^3$ with two unexpected cones of degrees 4 and 6 (note: $4 \cdot 6 = Z(F_4)$).

Projectivizing H_4 gives 60 points $Z(H_4) \subset \mathbb{P}^3$ with two unexpected cones of degrees 6 and 10 (note: $6 \cdot 10 = Z(H_4)$).

(see Wiśniewska-Zięba, arXiv:2107.08107).

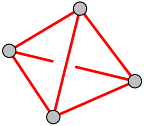
Teitler and a new recruit

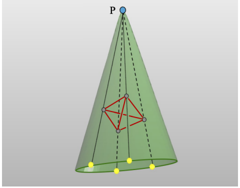
P. Wiśniewska

M. Zięba

Luca Chiantini's Observation: Levico Terme Workshop

As an example, consider 4 general gray points:





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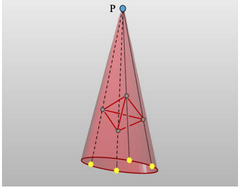
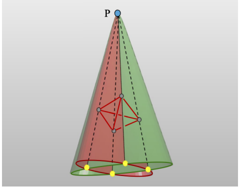


$=$



For any P, there are 2 cones through the 4 points with vertex P:

here a green cone of degree $a = 2$ through the $ab = 4$ gray points,

and a red cone of degree $b = 2$ through the $ab = 4$ gray points, and

the projection from P of the 4 points to the plane at the base of the cones is an (a, b)-complete intersection (i.e., the yellow points) in that plane.

Levico Terme Working Group, 2018

Alessandra Bernardi

Luca Chiantini

Graham Denham

Giuseppe Favacchio

Brian Harbourne

Juan Migliore

Tomasz Szemberg

Justyna Szpond

General projections to a complete intersection: Geproci

Definition: We say a finite set $Z \subset \mathbb{P}^3$ is (a, b)-*geproci* if its image $\overline{Z}$ under projection to a plane from a general point $P \in \mathbb{P}^3$ is an (a, b)-complete intersection.

Theorem (Levico Terme Working Group, 2018): A finite noncoplanar $Z \subset \mathbb{P}^3$ is (a, b)-geproci if $|Z| = ab$ and Z has cones of degrees $a \leq b$ with no common components.

Corollary: $Z(D_4)$, $Z(F_4)$ and $Z(H_4)$ are geproci!

Open Problems: (1) What other kinds of sets are geproci?

(2) If $Z \subset \mathbb{P}^3$ is noncoplanar and (a, b)-geproci with $3 \leq a \leq b$, Z has cones of degrees a and b with the same general vertex P. Are these cones unexpected?

(3) Does every noncoplanar geproci set Z have a nontrivial matroid (e.g., does it have subsets of 3 collinear points)?

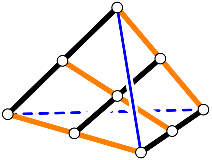
Some history: Polizzi and Panov

We know no interesting examples of geproci sets in $\mathbb{P}^n$ for $n > 3$.

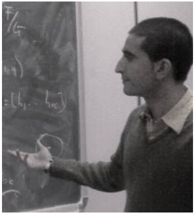
A geproci set Z in a plane $H \subset \mathbb{P}^3$ is called *degenerate*; it is just the complete intersection of two curves in H.

Question 1 (F. Polizzi 2011): Is every geproci set in $\mathbb{P}^3$ degenerate?

Answer (D. Panov, 2011): No!
Nondegenerate geproci sets are given by (a, b)-grids (i.e., sets Z of ab points where $Z = A \cap B$ and A consists of a skew lines and B consists of b skew lines).



Francesco Polizzi



and Dimitri Panov

We know more now: main POLITUS references

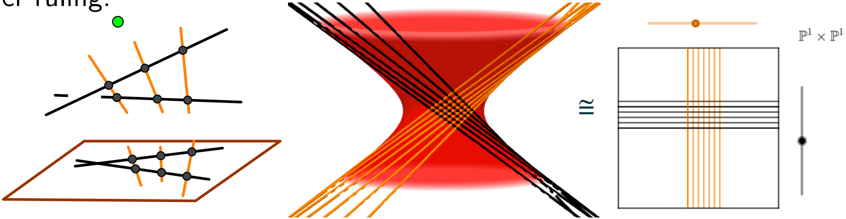
POLITUS 1: arXiv:2209.04820 POLITUS 2: arXiv:2308.00761

Łucja and Karolina Farnik, Tomasz Szemberg, Justyna Szpond, Luca Chiantini, Giuseppe Favacchio, Juan Migliore, Me.

Classifying geproci sets in $\mathbb{P}^3$ (POLITUS)

1. Degenerate case: two curves in a plane meeting transversely.

2. Grids: any set of b points on each of 2 skew lines, or the points where a lines in one ruling on a smooth quadric meet b lines in the other ruling.



3. Half grids: Deep connections with combinatorics!

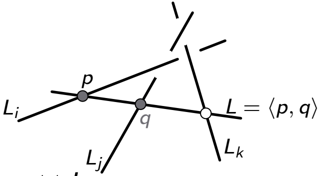
4. Everything else: still not well understood!

Case 3: Combinatorics (skew lines) = Geometry (geproci)

Given $\mathcal{L} = \{L_1, \ldots, L_s\}$, $s \geq 3$ skew lines in $\mathbb{P}^3$ (over any field).

Given $p \in L_i, q \in L_j, i < j$: Say $p \sim q$ iff

$$\langle p, q \rangle \cap L_k \neq \varnothing, \ k \neq i, j.$$



This generates an equivalence relation $\cong_{\mathcal{L}}$ on $\cup_i L_i$.
(Denote the equivalence class of a point p by $[p]$.)

Say Z has $(*)$ if $Z \subset \cup_i L_i$ satisfies $|Z \cap L_i| = r$ for all i where $r = |Z|/s$. Say a Z with $(*)$ is $(r, \mathcal{L})$-geproci if its projection $\overline{Z}$ from a general point to a plane is a complete intersection of type (r, s) with the image of $\mathcal{L}$ giving the generator of degree s.

Combinatorics = Geometry Theorem (POLITUS): Given Z with $(*)$. Then Z is $(r, \mathcal{L})$-geproci iff Z is a union of finite $\cong_{\mathcal{L}}$ equivalence classes.

New Results on the Combinatorics of Skew Lines.

Given $\mathcal{L} = \{L_1, \ldots, L_s\}$, $s \geq 3$ skew lines in $\mathbb{P}^3$ (over any field).

Let $p \in L_i$. There is a group $G_{\mathcal{L}} \subset \mathrm{Aut}(\mathbb{P}^1)$ such that $[p] \cap L_i$ is the $G_{\mathcal{L}}$-orbit of p.

Theorem: (POLITUS)

• $G_{\mathcal{L}}$ is abelian iff there are two or more lines (counted with multiplicity) meeting each line in $\mathcal{L}$.

• If $G_{\mathcal{L}}$ is infinite then all but at most two orbits are infinite, and the finite orbits are singletons. (When $G_{\mathcal{L}}$ is abelian, it acts by scale transformations or translations.)

Open Problem: Classify the finite groups that arise as $G_{\mathcal{L}}$.

Jake Kettinger and Allison Ganger

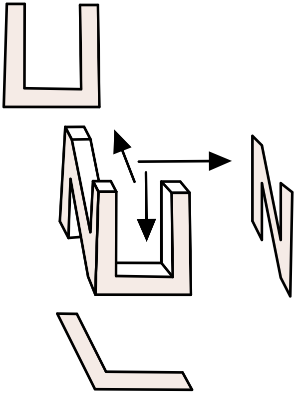
Jake has found a method to construct type 4 geproci sets in positive characteristics.

Allison Ganger has computed the group $G_{\mathcal{L}}$ for a range of cases.

Classifying geproci sets is an inverse scattering problem

Studying inverse scattering problems has led to remarkable advances in scientific knowledge. Here we propose carrying this idea over to classification problems in algebraic geometry.

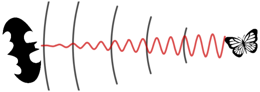
Inverse scattering Problems (ISP): try to discern structure from projected or reflected data.



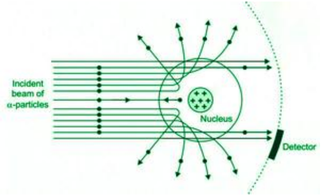
Idea: classify structures algebro-geometrically based on properties of projected images.

Some examples of ISP

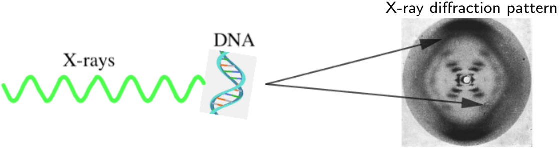
Echolocation (biology):



Rutherford scattering (physics; led to Bohr model of atom):



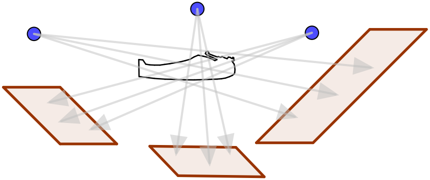
X-ray crystallography (chem/bio; led to DNA double helix model):



Rosalind Franklin
"Photo 51"

More examples

Tomography (medicine):



GePro-$\mathcal{P}$ (math): Pick a property $\mathcal{P}$ and classify finite point sets $Z \subset \mathbb{P}^n$ whose $\boxed{\text{Ge}}$neral $\boxed{\text{Pro}}$jections $\overline{Z}$ to a hyperplane satisfy $\boxed{\mathcal{P}}$.

Example 1: Say $\mathcal{P}$ means "$\overline{Z}$ is Gorenstein". Then a set Z of $n+1$ general points in $\mathbb{P}^n$ is gepro-$\mathcal{P}$ since the image $\overline{Z}$ is a set of $n+1$ general points in H, which is Gorenstein.

Open Problem 1: Classify gepro-Gorenstein sets Z.

Geproci is when $\mathcal{P}$ means "$\overline{Z}$ is a complete intersection". Every geproci set is also gepro-Gorenstein but not conversely.

Open Problem 2: Classify geproci sets in $\mathbb{P}^n$ for $n \geq 3$.

I migliori auguri a Sandro per 70 meravigliosi anni

e per tutti gli anni a venire!