Uwe's impact on some recent research

Brian Harbourne

Cather Professor of Mathematics
Fellow of the AMS
Department of Mathematics
University of Nebraska-Lincoln (UNL)

Uwe Fest: Conference on Commutative Algebra and its Interaction with Algebraic Geometry and Algebraic Combinatorics, August 12–16, 2024

Organizers: Juan Migliore, Sonja Petrović, Claudia Polini, Bernd Ulrich

August, 2024

Slides available eventually at my website (green text is clickable):
https://unlblh.github.io/BrianHarbourne/

$1/21 : 50 - 2.4 = 47.6$

A lot of my recent work traces to joint work with Uwe

So how did Uwe and I meet?

Meeting #1 (I think): 1992 Ravello conference
(pre-WWW so no photographic evidence!)

Here's a photo of the Villa Rufolo in Ravello:

2/21 : 50 − 4.8 = 45.2

Meeting #2 (also Italy), with photos:

Naples, 2000 (but no joint photo):

3/21 : 50 − 7.1 = 42.9

Meeting #3 (Italy, yet again): finally, a joint photo!

TonyFest, Acireale, Sicily, 2002

4/21 : $50 - 9.5 = 40.5$

Meeting #3 (Italy, yet again): finally, a joint photo!

TonyFest, Acireale, Sicily, 2002

5/21 : 50 − 11.9 = 38.1

My papers with Uwe

arXiv #	Journal & year	# authors: topic
arXiv:1404.4957	J. Algebra 2015	6: resurgences
arXiv:1508.00477	J. Alg. Comb. 2016	9: Waldschmidt consts.
arXiv:1502.00167	J. Algebra 2019	7: secant varieties
arXiv:1507.00380	TAMS 2017	4: matroids-symb. powers
arXiv:1602.02300	Compositio 2018	4: unexpectedness
arXiv:1805.10626	Mich. Math. J. 2021	4: root syst.-unexp
arXiv:2303.13317	Cortona Proc., to app	3: survey on unexp

I gave a 2018 Notre Dame algebra seminar on arXiv:1602.02300:

Matthew Dyer (of Notre Dame) pointed out one of our examples was a projectivized root system! That led to:

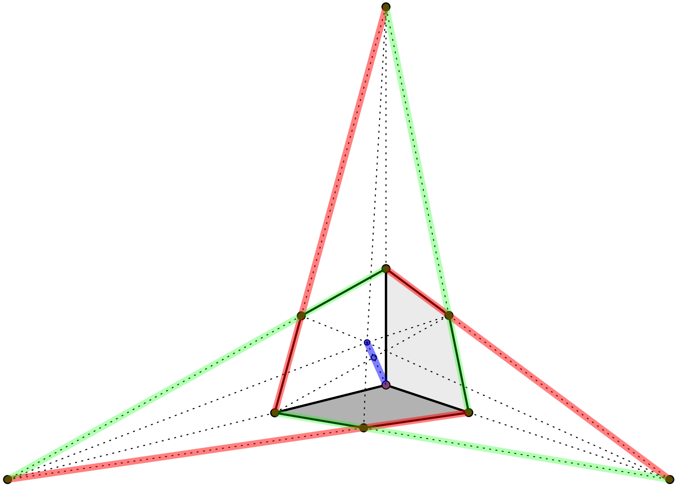
Mich. Math. J. 2021: Harbourne, Migliore, Nagel, Teitler (HMNT). "Unexpected hypersurfaces and where to find them,"

which has led to a new paradigm in algebraic geometry!

$6/21 : 50 - 14.3 = 35.7$

The HMNT example that led to the new paradigm

The 12 point projectivization $Z = Z_{D_4}$ of the D_4 root system!



7/21 : 50 − 16.7 = 33.3

HMNT: Z_{D_4} has two unexpected cones (TL;DR)

- $Z = Z_{D_4} \subset \mathbb{P}^n$: $n = 3$, $|Z| = r = 12$ points.

- Z imposes indep conditions on forms H_d of degree d, hence:

$$\dim[I_Z]_d = \begin{cases} 0 & d < 3 \\ \binom{d+n}{n} - r & d \geq 3. \end{cases}$$

- exp dim of all $H_{d,m} \in [I_Z]_d$ vanishing at general point P with multiplicity m:

$$\exp\dim = \max\{0, \text{virtual dim}\} = \max\left\{0, \dim[I_Z]_d - \binom{n+m-1}{n}\right\}.$$

- For cones $H_{d,m}$ we have $m = d$:

d	virtual dim	actual dim	conclusion
3	-2	1	this cubic cone is unexpected
4	3	4	the quartic cones are unexpected.

8/21 : 50 − 19.0 = 31.0

HMNT: Z_{D_4} has two unexpected cones (short version)

- P is a general point
- $Z_{D_4} \subset C_3$ where C_3 is a cubic cone with vertex P
- $Z_{D_4} \subset C_4$ where C_4 is a quartic cone with vertex P
- $C_3 \cap C_4 =$ the cone over Z_{D_4} of 12 lines through P

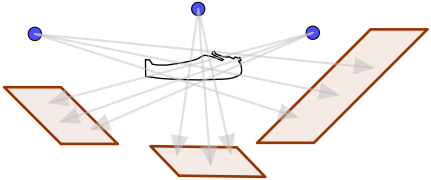
Where do these cones come from?

So what? (first noticed at Levico Terme conference, 2018): This means Z_{D_4} projects from P to a complete intersection! I.e., Z_{D_4} is geproci (this was the first "nontrivial" example; in this case Z_{D_4} is a "half grid").

The Paradigm: This is an example of applying an inverse scattering perspective to algebraic geometry; i.e., study point sets by the properties of their projections.

9/21 : 50 − 21.4 = 28.6

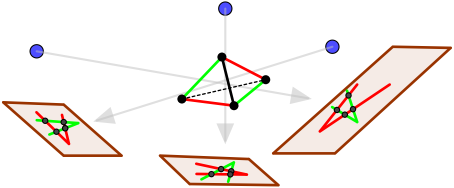
Tomography: an inverse scattering example



Apply Inverse Scattering perspective in Algebraic Geometry:

GePro-$\mathcal{P}$: Pick a property $\mathcal{P}$ and classify finite point sets $Z \subset \mathbb{P}^n$ whose $\boxed{\text{Ge}}$neral $\boxed{\text{Pro}}$jections $\overline{Z}$ to a hyperplane H satisfy $\boxed{\mathcal{P}}$.

Example: Geproci (i.e., $\mathcal{P}$ means: $\overline{Z}$ is a complete intersection).



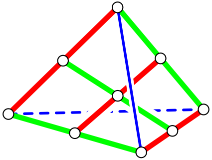
11/21 : 50 − 26.2 = 23.8

Trivial examples of finite sets Z that are geproci

If Z is contained in a hyperplane and already a complete intersection, then it is geproci.

If $Z \subset \mathbb{P}^2$, then Z is geproci.

"Grids" in $\mathbb{P}^3$ are geproci.
(A grid Z consists of ab points which are the intersection of space curves of degrees a and b, each curve consisting of lines).



Open Problem: Find $n > 3$ (if any) having nondegenerate examples of geproci $Z \subset \mathbb{P}^n$.

We know examples only for $n = 3$, in which case we say Z is (a, b)-geproci if $\overline{Z}$ is an (a, b) complete intersection.

12/21 : 50 − 28.6 = 21.4

An example of nondegenerate nongrid geproci: Half grids

Half grids: A half grid Z is an (a, b)-geproci set with b points of Z on each of a skew lines, but Z is not a grid, so there is no separate set of b skew lines with a points on each line. (For example, Z_{D_4} is $(4, 3)$-geproci, not a grid, and contained in a quartic of $a = 4$ skew lines with $b = 3$ points on each line.)

(**POLITUS**: Luca Chiantini, Łucja Farnik, Giuseppe Favacchio, Brian Harbourne, Juan Migliore, Tomasz Szemberg, Justyna Szpond)

Theorem (POLITUS): For every $n \geq 3$, there is an $(n, n+1)$-geproci half grid of n points on each of $n+1$ skew lines (which POLITUS calls the "standard construction"; e.g., for $n = 3$, this gives the unique half grid, which is Z_{D_4}).

Theorem (Jake Kettinger): For any finite field F, let $|F| = q$. Then $Z = \mathbb{P}^3_F \subset \mathbb{P}^3_{\overline{F}}$ is a half grid of $q + 1$ points each on $q^2 + 1$ skew lines which come from a kind of "Hopf fibration". E.g., if $q = 3$, Z is a half grid of 4 points each on 10 lines.

13/21 : 50 − 31.0 = 19.0

Combinatorial structure of half grids

Theorem (POLITUS): Say Z is a half grid of a points on each of $b > 3$ lines $L_1, \ldots, L_b$. Then $Z' = Z \cap (L_{i_1} \cup L_{i_2} \cup L_{i_3})$, $i_1 < i_2 < i_3$, is a grid. In particular, for each point $p_1 \in Z \cap L_{i_1}$ there are points $p_2 \in Z \cap L_{i_2}$ and $p_3 \in Z \cap L_{i_3}$ such that p_1, p_2, p_3 are collinear.

Open Question: When are finitely many skew lines the lines of a half grid?

A finite set of 3 or more skew lines always has a "groupoid" structure (as will see).

Corollary (POLITUS): The lines $L_1, \ldots, L_b$ have a finite groupoid structure, and Z is a union of orbits of this finite groupoid.

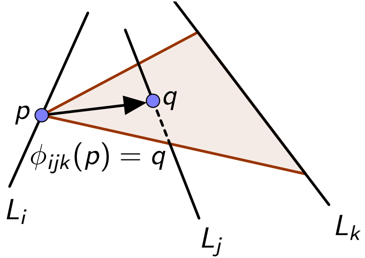
Open Question: When does a finite set of skew lines have a finite groupoid?

14/21 : 50 − 33.3 = 16.7

Combinatorics of skew lines: groupoids

Groupoid: A category $\mathcal{G}$ whose arrows all are invertible.

Example: Skew lines $\mathcal{L} = \{L_1, \ldots, L_r\}$, $r \geq 3$, give a groupoid $\mathcal{G}_{\mathcal{L}}$.
The lines L_i are the Objects. Define arrows $\phi_{ijk} : L_i \xrightarrow{L_k} L_j$:



Then $\mathrm{Hom}(L_i, L_j) =$ all possible compositions $\phi_{j_s j k_{s+1}} \cdots \phi_{j_1 j_2 k_2} \phi_{i j_1 k_1}$.

Note: $\mathrm{Hom}(L_i, L_i)$ is a group, independent of i, the group of the groupoid. The groupoid is finite if and only if the group of the groupoid is.

Open Problem: When is the group finite?

15/21 : 50 − 35.7 = 14.3

Groupoid orbits, geproci half grids and the Hopf fibration

The groupoid $\mathcal{G}_{\mathcal{L}}$ acts on points of the skew lines $\mathcal{L} = \{L_1, \ldots, L_r\}$, so we can talk about groupoid orbits.

Theorems (Allison Ganger):

(1) If F is a finite field, then the points $Z = \mathbb{P}^3_F \subset \mathbb{P}^3_{\overline{F}}$ form a single groupoid orbit on the skew lines coming from the "Hopf fibration".

(2) Up to projective equivalence, the half grid lines of the standard construction can be chosen to be fibers of the "Hopf fibration" and then the half grid points form a single groupoid orbit on these lines.

So what is this "Hopf fibration"?

A visualization of the traditional Hopf fibration $S^3 \to S^2$

16/21 : 50 − 38.1 = 11.9

The Hopf fibration algebraically

The original Hopf fibration comes from the field extension $\mathbb{R} \subset \mathbb{C}$:

$$S^3 \to \mathbb{P}^3_{\mathbb{R}} = \mathbb{P}_{\mathbb{R}}(\mathbb{C} \oplus \mathbb{C}) \to \mathbb{P}_{\mathbb{C}}(\mathbb{C} \oplus \mathbb{C}) = \mathbb{P}^1_{\mathbb{C}} = S^2.$$

More generally: let $F \subset K$ be any degree 2 field extension. Then:

- K is a 1 dimensional K and a 2 dimensional F vector space;
- $K \oplus K$ is a 2 dimensional K vector space;
- $K \oplus K$ is a 4 dimensional F vector space;

and we get a canonical "Hopf fibration" map

$$\mathbb{P}^3_F = \mathbb{P}_F(K \oplus K) \to \mathbb{P}_K(K \oplus K) = \mathbb{P}^1_K$$

where the fibers are collinear sets of points defining skew lines.

Theorem (Ganger): When $F \subset K$ is a degree 2 extension of finite fields, the group of the groupoid on the fibers of the "Hopf fibration" is K^*/F^*, hence cyclic of order $\frac{|K^*|}{|F^*|} = \frac{|F|^2-1}{|F|-1} = |F| + 1$.

17/21 : 50 − 40.5 = 9.5

More combinatorics

Consider $\mathbb{P}^3_F$ over a finite field F. In combinatorics, skew lines $L_1, \ldots, L_r$ in $\mathbb{P}^3_F$ with each L_i defined over F is called a *spread*.

If every point of $\mathbb{P}^3_F$ is in some line it is a *full* spread, otherwise a partial spread.

A spread $L_1, \ldots, L_r$ is *maximal* if every F-line L meets some line L_i.

Problems partially addressed by combinatorists:
Count the number of full spreads up to projective equivalence. (The "Hopf fibration" always gives one; usually there are others. Hence $Z = \mathbb{P}^3_F$ is usually a half grid in more than one way.)

More generally, count the number of maximal spreads up to projective equivalence.

Problems not yet addressed by combinatorists:
Study the groupoid for maximal spreads. For example, when is the group nonabelian? (Ganger gives nonabelian examples.)

18/21 : 50 − 42.9 = 7.1

Nondegenerate nongrid non-half grid geproci sets

Very few examples are known in characteristic 0:

(1) A $(6, 10)$-geproci from the H_4 root system (Fraś and Zięba).

(2) A $(5, 8)$-geproci (arxiv:2209.04820).

(3) A $(10, 12)$-geproci (arxiv:2209.04820).

Open Problem: Are there more examples in characteristic 0?

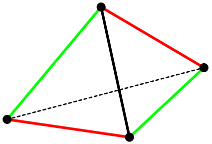
Open Problem: For each $r \gg 0$, construct a nondegenerate nongrid non-half grid geproci $Z_r \subset \mathbb{P}^3_{\mathbb{K}}$ with $|Z_r| \geq r$, or show this is impossible.

Note: Jake Kettinger has settled this affirmatively in positive characteristics.

19/21 : 50 − 45.2 = 4.8

Some open problems

A $(2, 2)$-grid is a nontrivial geproci set of 4 linearly general points:



No other nontrivial geproci set that we know of is linearly general.

Open problem: Find a nontrivial linearly general geproci set or prove none exist.

Example: Say $\mathcal{P}$ means "$\overline{Z}$ is Gorenstein". Then a set Z of $n + 1$ general points in $\mathbb{P}^n$ is gepro-$\mathcal{P}$ since the image $\overline{Z}$ is a set of $n + 1$ general points in a hyperplane, which is Gorenstein.

Open Problem: Classify gepro-Gorenstein sets Z.

Every geproci set is also gepro-Gorenstein but not conversely.

20/21 : 50 − 47.6 = 2.4

Thanks for your attention! And thanks to Uwe for his many years of contributions to mathematics!

MFO
1999

MFO
2009

Cortona
2022

21/21 : 50 — 50.0 = 0.0