# Rational Amusements to Lighten a Long Year of Social Distancing

Brian Harbourne

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Colloquium, University of Arkansas: April 8, 2021

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**Abstract**: In 1821 John Jackson published *Rational amusement for winter evenings or, A collection of above 200 curious and interesting puzzles and paradoxes*. Some of these puzzles foreshadowed open problems in combinatorics about line arrangements, which have recently become relevant to a growing body of work in commutative algebra and algebraic geometry. I will describe some of this work and its history and discuss how it relates to an old open problem in algebraic geometry, called the Bounded Negativity Conjecture, and to a newer problem in commutative algebra called the Containment Problem. No background in algebraic geometry, commutative algebra or combinatorics will be assumed.

# Jackson's cover page:

RATIONAL AMUSEMENT

FOR

*Winter Evenings;*

OR,

A Collection of above

200 CURIOUS AND INTERESTING

PUZZLES AND PARADOXES

*relating to*

ARITHMETIC, GEOMETRY, GEOGRAPHY, &c.

WITH THEIR SOLUTIONS,

AND FOUR PLATES.

---

*DESIGNED CHIEFLY FOR YOUNG PERSONS.*

---

BY

JOHN JACKSON,

*Private Teacher of the Mathematics.*

---

London:

SOLD BY LONGMAN, HURST, REES, ORME, AND BROWN;
G. AND W. B. WHITTAKER; AND
HARVEY AND DARTON.
AND BARRY AND SON, HIGH-STREET, BRISTOL.

1821.

# Jackson's cover page (revised):

**RATIONAL AMUSEMENT**

FOR

*Evenings in Quarantine;*

OR,

A Collection of above

200 CURIOUS AND INTERESTING

**PUZZLES AND PARADOXES**

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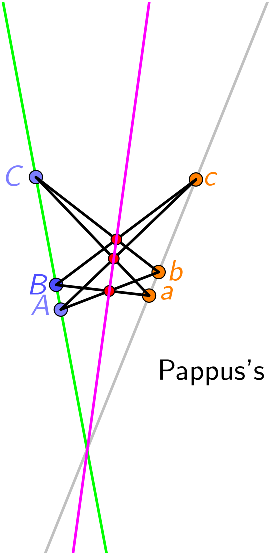
1821.

# Orchard Problems

1. Your aid I want, nine trees to plant
In rows just half a score;
And let there be in each row three.
Solve this: I ask no more.

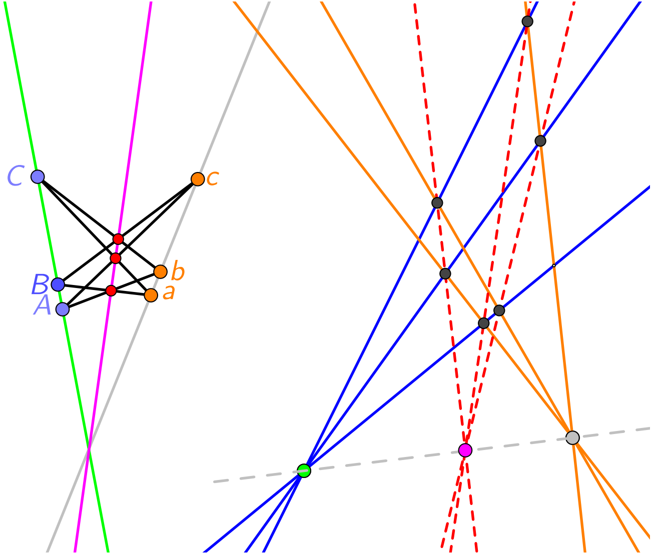
2. Fain would I plant a grove in rows,
But how must I its form compose
With three trees in each row;
To have as many rows as trees;
Now tell me, artists, if you please;
'Tis all I want to know.

# Orchard Problem #2 (9 trees, 9 rows of three)



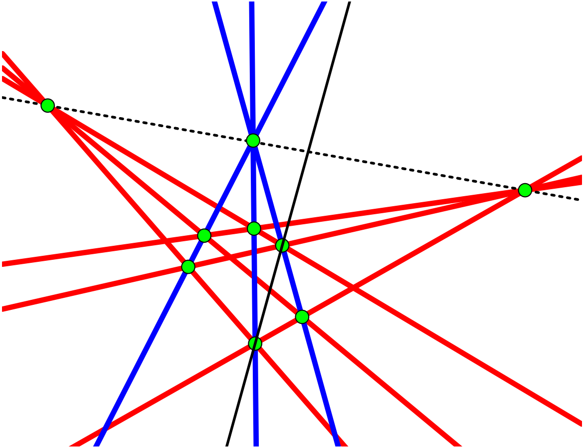

Pappus's hexagon theorem (320 AD)

# Orchard Problem #1 (9 trees, 10 rows of 3): Projective Duality



# Let there be in each row three: but what's a row?

Suppose we define a row to be any line defined by two trees. Then some rows have only two trees:

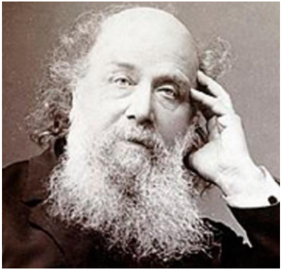


# Sylvester Reformulates

**J.J. Sylvester's Reformulation**: Find a finite set of noncolinear points with no "ordinary lines" (i.e., no lines which contain exactly two of the points).

**Conjecture** (J.J. Sylvester, 1893, The Educational Times, 46 (383): 156): It can't be done!

> **11851.** (Professor SYLVESTER.) — Prove that it is not possible to arrange any finite number of real points so that a right line through every two of them shall pass through a third, unless they all lie in the same right line.



# Sylvester's Orchard Problem (equivalent dual version)

**Conjecture** (original): If for a collection of $d$ points there are no lines through exactly two points, then the points are colinear.

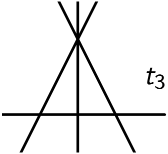
**Conjecture** (dual): If for a collection of $d$ lines there are no points where exactly two lines cross, then the lines are concurrent.

**Definition**: Given an arrangement of $d$ lines, for $k \geq 2$:

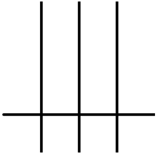
$$t_k = \text{number of points where exactly } k \text{ lines cross}$$
$$= \text{number of points of multiplicity } k$$

(including points at infinity when some of the lines are parallel)

**Example**:



$t_3 = 1,\ t_2 = 3$



**Contrapositive of dual**: If $d$ lines are not concurrent, then $t_2 > 0$.

# Melchior's 1941 Solution of Dual to Orchard Problem

**Theorem**: A non-concurrent real arrangement $L$ of lines satisfies

$$t_2 \geq 3 + \sum_{k \geq 3} (k-3) t_k$$

or

$$t_2 = 3 + \sum_{k \geq 3} (k-3) t_k + \Delta_L$$

for some $\Delta_L \geq 0$.

**E. Melchior**: *Über Vielseite der projektiven Ebene. Deutsche Math., 5 (1941) 461–475*

# Complex line arrangements

Consider sets $L$ of $d$ lines $ax + by + c = 0$ in $\mathbb{C}^2$ (so $a, b, c \in \mathbb{C}$).

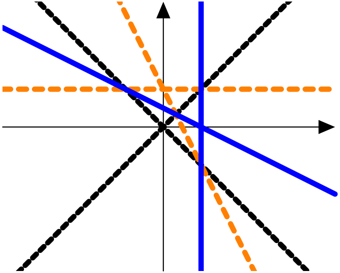
**Open Problem**: Classify all complex line arrangements with $t_2 = 0$.

Only four types are known:

(1) $d \geq 3$ concurrent lines: $t_d = 1$

(2) the $n \geq 3$ Fermat arrangements: defined by the factors of $f_L = (x^n - y^n)(x^n - (x + y - 1)^n)(y^n - (x + y - 1)^n)$:

$d = 3n, t_3 = n^2, t_n = 3$



$n = 2$

$t_n = 1 + 1 + 1 = 3$ and $t_3 = n^2$

# Two more are known

(3) an example due to F. Klein (1879):
$d = 21, t_3 = 28, t_4 = 21$

(4) and an example due to A. Wiman (1896):
$d = 45, t_3 = 120, t_4 = 45, t_5 = 36$

# Open Problem

- Classify configurations with $t_2 = 0$ over $\mathbb{C}$.

It would be very interesting to know if there are any other complex line arrangements with $t_2 = 0$, since these have recently become important in Algebraic Geometry in studying *bounded negativity* and in Commutative Algebra in studying *symbolic powers*.

# $H$ constants (Oberwolfach 2010) and Bounded Negativity

The $H$ constant $H(L)$ of a line arrangement $L$ with $d_L$ lines is

$$H(L) = \frac{d_L - \sum_{k>1} t_k k}{\sum_{k>1} t_k}.$$

**Question**: How negative can $H(L)$ be?

**Remark**: You can define $H(C)$ for any singular complex reduced plane algebraic curve $C$. A version of the 100 year old still open Bounded Negativity Conjecture is that $H(C)$ cannot be arbitrarily negative. For example, no singular complex reduced irreducible $C$ is known with $H(C) \leq -2$.

But what can we say when $C$ is a union of lines? I.e., when $C$ is a line arrangement $L$?

# $H$-constants for Line Arrangements

**Theorem** (arXiv 1407.2966): Let $L$ be a real line arrangement. Then $H(L) > -3$ and there is a sequence $L_1, L_2, \ldots$ of real line arrangements such that $H(L_n) \overset{n\to\infty}{\longrightarrow} -3$.

**Proof**: Case 1. For concurrent line arrangements $L$: $H(L) = 0$.

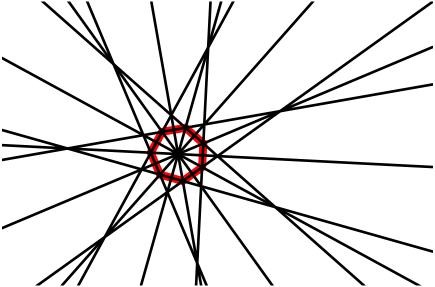
Case 2. For non-concurrent line arrangements $L$, use Melchior $t_2 = 3 + \sum_{k\geq 3}(k-3)t_k + \Delta_L$.

$H(L) = \frac{d-\sum_{k\geq 2} t_k k}{\sum_{k\geq 2} t_k}$

$$> -\frac{\sum_{k\geq 2} t_k k}{\sum_{k\geq 2} t_k} = -\frac{2(3+\Delta_L) + 3\sum_{k\geq 3} t_k(k-2)}{3+\Delta_L+\sum_{k\geq 3} t_k(k-2)} > -3$$

# Proof cont.

For $L_n$ take the $d = 2n$ lines given by the $n$ sides of a regular $n$-gon with odd $n$ and the $n$ angle bisectors.



$L_7$

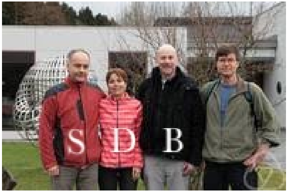
Then there are:
$t_2 = n$ points of multiplicity 2,
$t_3 = \binom{n}{2}$ points of multiplicity 3, and
$t_n = 1$ point of multiplicity $n$, giving

$$H(L_n) = \frac{d - \sum_{k\geq 2} t_k k}{\sum_{k\geq 2} t_k} = -3 + \epsilon_n \overset{n\to\infty}{\longrightarrow} -3.$$

# Analogous result over $\mathbb{C}$

**Theorem**: Let $L$ be a complex line arrangement. Then $H(L) > -4$.

T. Bauer, S. Di Rocco, B. Harbourne, J. Huizenga, A. Lundman, P. Pokora, T. Szemberg, *Bounded Negativity and Arrangements of Lines*, IMRN (2015) 9456–9471





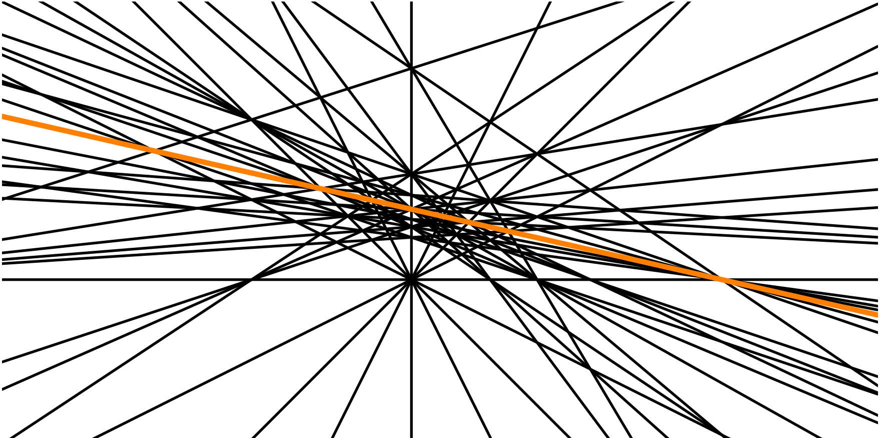
**Comment**: The most negative $H(L)$ known is for the Wiman arrangement, which gives $H(L) = -\frac{225}{67} \approx -3.36$.

# More Open Problems

(1) What is the minimum $H(L)$ for a complex line arrangement $L$? Note Wiman has $t_2 = 0$; are there additional line arrangements with $t_2 = 0$?

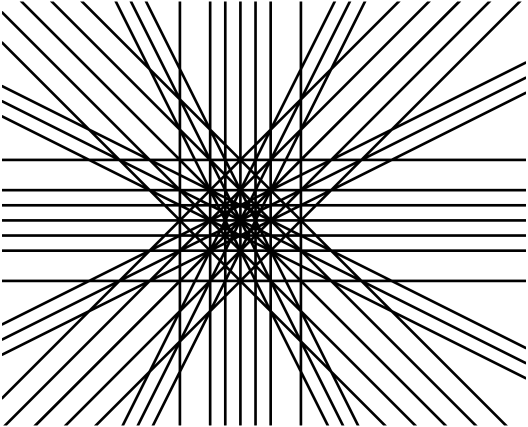
(2) What is minimum $H(L)$ for rational line arrangements $L$? Best known is following, with $d = 37$,

$t_2 = 72, t_3 = 72, t_4 = 24, t_6 = 10, t_8 = 3$ and $H = \frac{-503}{181} \approx -2.779$:



# Redo

Here's the result of moving the orange line off to infinity:



# The Containment Problem and the Chudnovsky Problem

Line arrangements are relevant to some more recent work in Comm Alg. Here, for example, are two relevant recent papers:

M. Johnson: Containing symbolic powers in regular rings, Comm. Algebra 42:8 (2014) 3552–3557

L. Fouli, P. Mantero and Y. Xie: Chudnovsky's conjecture for very general points in $\mathbb{P}^n$, J. Algebra 498 (2018) 211–227

These two problems are related!

# Fundamental Question in Hermite Interpolation

Let $p_1, \ldots, p_r \in \mathbb{C}^n$ and let $mZ = mp_1 + \cdots + mp_r$.

Let $I(p_i) = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f(p_i) = 0\}$ and

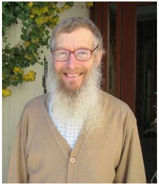
$$I(mZ) := I(p_1)^m \cap \cdots \cap I(p_r)^m \overset{CRT}{=} (I(p_1) \cdots I(p_r))^m \overset{CRT}{=} I(Z)^m.$$

**Question**: What is the least degree $\alpha(I(mZ))$ for $0 \neq f \in I(mZ)$?

**Definition** (Waldschmidt Constant): $\widehat{\alpha}(I(Z)) := \lim_{m\to\infty} \frac{\alpha(I(mZ))}{m}$

**Theorem** (W-S, 1977): $$\frac{\alpha(I(Z))}{n} \leq \widehat{\alpha}(I(Z)) \leq \frac{\alpha(I(mZ))}{m}$$

M. Waldschmidt, *Propriétés arithmétiques de fonctions de plusieurs variables II*, Séminaire P. Lelong (Analyse), LNM 578 (1977) 108–135.



H. Skoda, *Estimations $L^2$ pour l'opérateur $\overline{\delta}$ et applications arithmétiques*, LNM 578 (1977) 314–323.

# Homogenization and Symbolic Powers

Given $f \in \mathbb{C}[x_1, \ldots, x_n]$, $h_f \in \mathbb{C}[x_0, \ldots, x_n]$ is homogeneous:

$$\text{if } f = 7x_1^4 + x_2x_3 + 2,$$

then its homogenization is

$$h_f = 7x_1^4 + x_2x_3x_0^2 + 2x_0^4.$$

And given any ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, we define

$$h_I = (h_f : f \in I) \subseteq \mathbb{C}[x_0, \ldots, x_n].$$

This is a homogeneous ideal.

Define $(h_{I(Z)})^{(m)}$ to be $h_{I(Z)^m} = h_{I(mZ)}$.

**Containment Problem**: For which $r$ and $m$ is $(h_{I(Z)})^{(m)} \subseteq (h_{I(Z)})^r$?

# The ELS-HH Containment Theorem

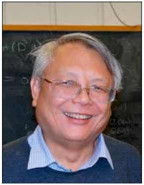
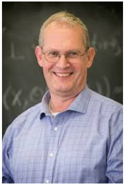
**Theorem** (ELS-HH) Let $Z = p_1 + \cdots + p_r$ for $p_i \in \mathbb{C}^n$. Then

$$(h_{I(Z)})^{(m)} \subseteq (h_{I(Z)})^r$$

for $m \geq nr$.

L. Ein, R. Lazarsfeld, K. Smith, *Uniform Bounds and Symbolic Powers on Smooth Varieties*, Invent. Math. 144 (2001), 241–252.

M. Hochster, C. Huneke, *Comparison of symbolic and ordinary powers of ideals*, Invent. Math. 147 (2002), 349–369.


Note: Mark Johnson's paper gives a generalization of this result.

# Comments

Bocci-Harbourne (2010): The containment $(h_{I(Z)})^{(m)} \subseteq (h_{I(Z)})^r$ for $m \geq nr$ is optimal:

for any $c < n$ there is a $Z$ and $m$ and $r$ with $m > cr$ but

$$(h_{I(Z)})^{(m)} \not\subseteq (h_{I(Z)})^r.$$

(Harbourne (2009)): The Waldschmidt-Skoda bound

$$\frac{\alpha(I(Z))}{n} \leq \widehat{\alpha}(I(Z))$$

is an easy consequence of the ELS-HH containment theorem.

# Chudnovsky Conjecture

**Conjecture** (Chudnovsky, 1980): $\frac{\alpha(I(Z))+n-1}{n} \leq \widehat{\alpha}(I(Z))$

If true, this is sharp (due to line arrangements and hyperplane arrangements).

(Trivial for $n = 1$; proved by Chudnovksy for $n = 2$.)

G.V. Chudnovsky, *Singular points on complex hypersurfaces and multidimensional Schwarz Lemma*, M.-J. Bertin (Ed.), Séminaire de Théorie des Nombres Delange-Pisot-Poitou, Paris, 1979-80, Prog. Math., vol. 12, Birkhäuser.

Note: The paper of Fouli-Mantero-Xie proves the conjecture when the points of $Z$ are sufficiently general.

# Reconsidering the optimality of ELS-HH containment

Let $Z = p_1 + \cdots + p_r \subset \mathbb{C}^n$, $J = h_{I(Z)} \subset \mathbb{C}[x_0, \ldots, x_n]$. By EHS-HH, $J^{(mn)} \subseteq J^m$. We can improve this by making $J^m$ smaller or $J^{(mn)}$ bigger. First let's try making $J^m$ smaller.

**Conjecture** (H.-Huneke, 2013): Let $M = (x_0, \ldots, x_n) \subset K[x_0, \ldots, x_n]$. Let $Z = p_1 + \cdots + p_r$, $J = h_{I(Z)}$. Then

$$J^{(mn)} \subseteq M^{mn-m} J^m.$$

B. Harbourne and C. Huneke, Are symbolic powers highly evolved?, J. Ramanujan Math. Soc. 28 (2013), 311–330

Notes:

1. This conjecture implies the Chudnovsky Conjecture.

2. Fouli-Mantero-Xie prove the conjecture above in certain cases.

# Now let's try making $J^{(mn)}$ bigger.

It's easy to see $J^{(mn-n)} \subseteq J^m$ can fail.

What about $J^{(mn-n+1)} \subseteq J^m$?

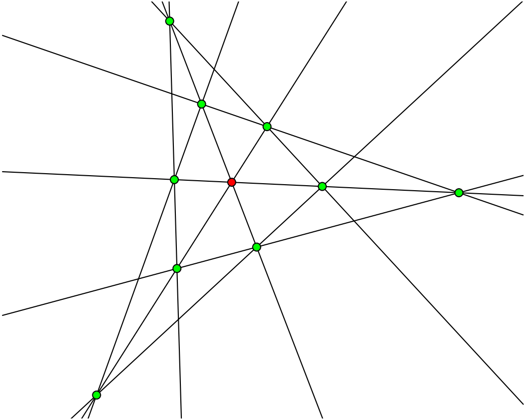
**Open Problem**: Classify $Z$ with $J^{(mn-n+1)} \subseteq J^m$.

No failures are known with $n > 2$ or $m > 2$. And all known failures for $n = m = 2$ come from line arrangements with $t_2$ "small". In particular, all known nontrivial complex arrangements with $t_2 = 0$ give counterexamples for $m = 2$!

A noncontainment example defined over $\mathbb{Q}$ with $t_2$ small can be given based on an orchard arrangement, thereby taking us back to the beginning!

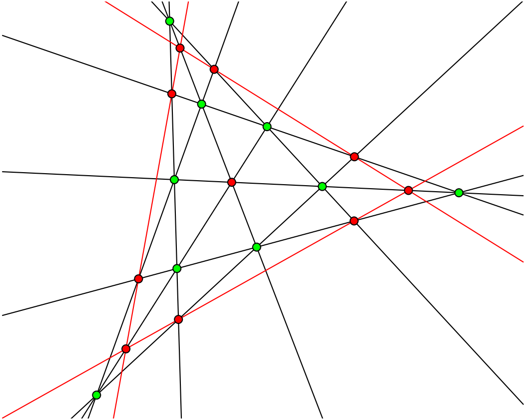
# The example starts with 9 trees in 9 rows of three:

Start by adjusting a dual Pappus orchard arrangement:



# The example

Start by adjusting a dual Pappus orchard arrangement:

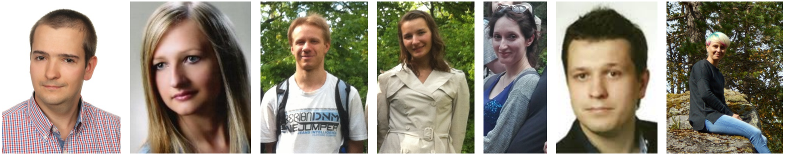


Add 3 lines to get $t_2 = 9$ and $t_3 = 19$ (so $t_2$ is "small"). Take $Z$ to be just the 19 triple points. Let $J = h_{I(Z)}$.

Then $J^{(3)} \not\subseteq J^2$. (First done over $\mathbb{R}$ but also works over $\mathbb{Q}$.)

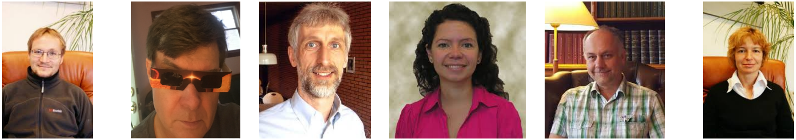
# The example

The real case:



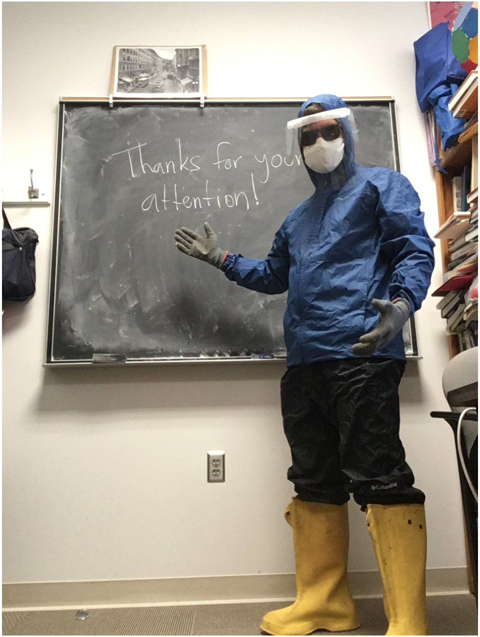
arXiv:1310.0904: A. Czapliński, A. Główka-Habura, G. Malara, M. Lampa-Baczyńska, P. Łuszcz-Świdecka, P. Pokora, J. Szpond.

The rational case:

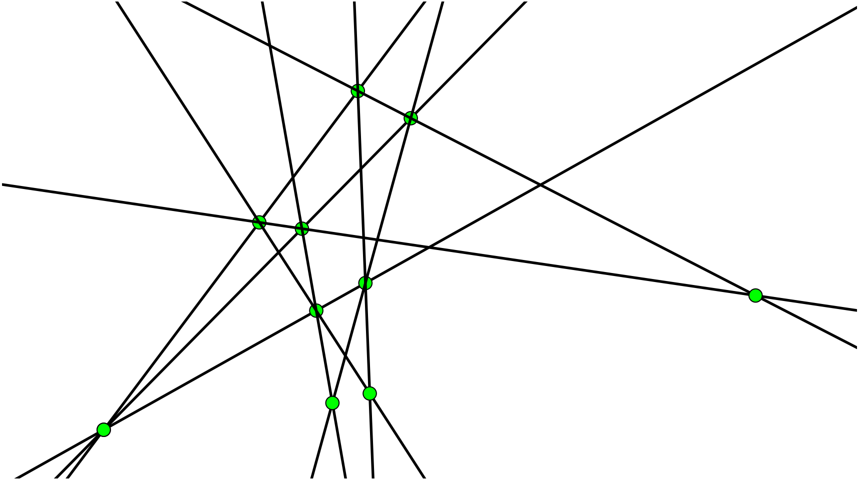


arXiv:1404.4957: M. Dumnicki, B. Harbourne, U. Nagel, A. Seceleanu, T. Szemberg, H. Tutaj-Gasińska.

# Pandemic 2020-2021: All suited up and ready to teach ...



# Orchard Problem #2: Solution 2 (10 trees, 10 rows of 3)



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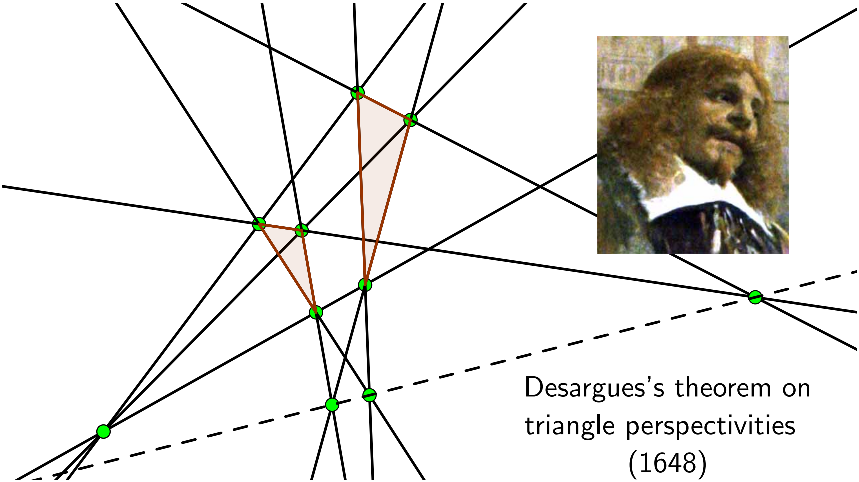

Desargues's theorem on triangle perspectivities (1648)